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ON EQUIVALENCE OF KELVIN AND MAXWELL MULTIELEMENT MODELS

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Abstract. *The problem of Kelvin and Maxwell model equivalence in one- and three-dimensional cases is investigated. The equations of one model transfer to the other one have been given.*

Key words: *linear viscoelasticity, Kelvin model, Maxwell model, equivalent models.*

ОБ ЭКВИВАЛЕНТНОСТИ МНОГОЭЛЕМЕНТНЫХ МОДЕЛЕЙ КЕЛЬВИНА И МАКСВЕЛЛА

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Аннотация. *Исследован вопрос об эквивалентности моделей Кельвина и Максвелла в одномерном и трехмерном случаях. Приведены формулы перехода от одной модели к другой.*

Ключевые слова: *линейная вязкоупругость, модель Кельвина, модель Максвелла, эквивалентные модели.*

ПРО ЕКВІВАЛЕНТНІСТЬ БАГАТОЕЛЕМЕНТНИХ МОДЕЛЕЙ КЕЛЬВІНА ТА МАКСВЕЛЛА

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Анотація. *Досліджено питання про еквівалентність моделей Кельвіна і Максвелла в одновимірному і тривимірному випадках. Наведено формули переходу від однієї моделі до іншої.*

Ключові слова: *лінійна в'язкопружність, модель Кельвіна, модель Максвелла, еквівалентні моделі.*

Introduction

In engineering practice of modelling the physical structure of materials with linear visco-elastic properties the mechanical models composed of elements of elasticity and viscosity interconnected in different ways have become rather widespread. In [1], some (three or four-element) one-dimensional models were observed to be equivalent at the expense of selecting component characteristics. This means that the relation between the stress and strain generated by these models is equal.

The authors of [2] suggested a hypothesis of the equivalence of Kelvin and Maxwell multiple-unit models.

In the paper we substantiated this statement for one-dimensional models, and the conditions under which it was true for a three-dimensional case.

Objective and Problem Setting

In Fig. 1–4 and 5–8 the structural diagrams of models called Kelvin and Maxwell generalized models in the scientific literature respectively, are shown:

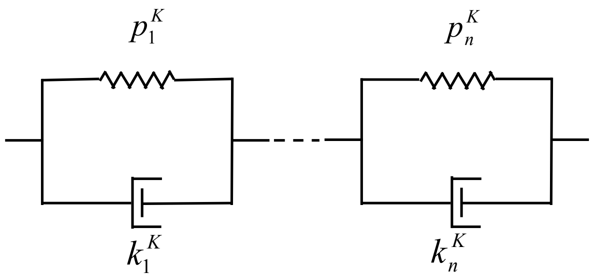

Fig. 1. Kelvin generalized model with missing individual elements of stiffness and viscosity

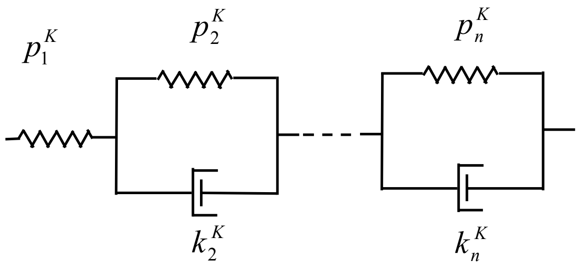

Fig. 2. Kelvin generalized model with an individual stiffener

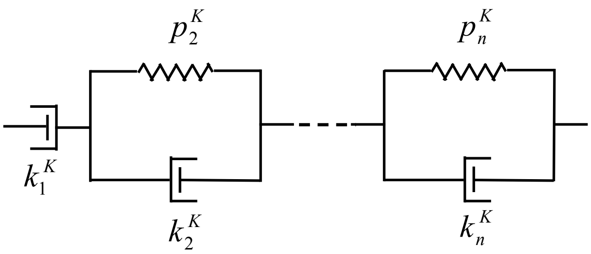

Fig. 3. Kelvin generalized model with an individual viscosity element

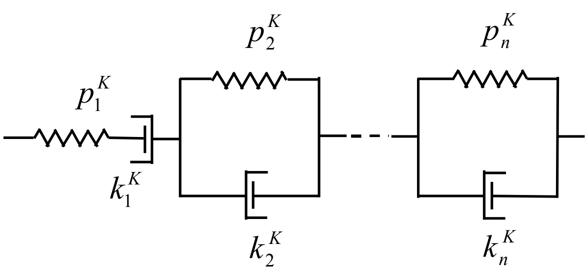

Fig. 4. Kelvin generalized model with some individual elements of stiffness and viscosity

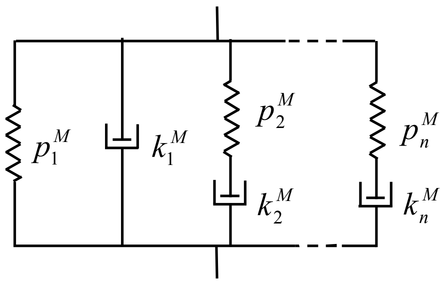

Fig. 5. Maxwell generalized model with some individual elements of stiffness and viscosity

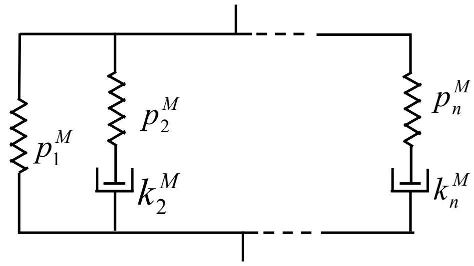

Fig. 6. Maxwell generalized model with an individual stiffener

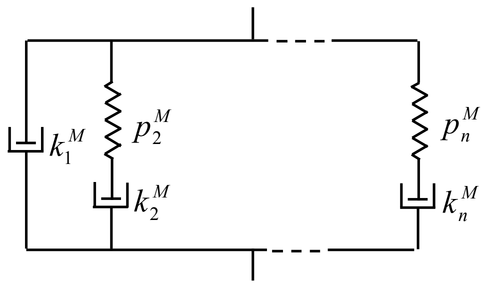

Fig. 7. Maxwell generalized model with an individual viscosity element

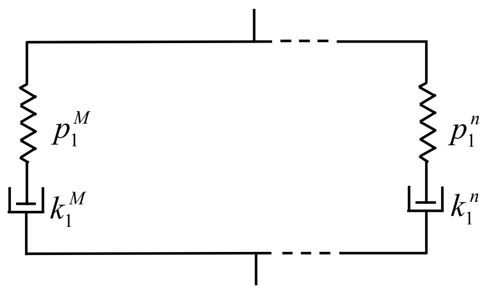

Fig. 8. Maxwell generalized model with missing individual elements of stiffness and viscosity

The main aim of the work is to investigate the question: under what condition for the model of one type there exists an equivalent model of the other type?

One-dimensional models

In one-dimensional case for the mechanical models of the theory of linear viscoelasticity the relation between the stress (σ) and the strain (ε) can be seen in terms of the differential equation (hereinafter referred to as the model equation), set up by certain rules [1], [3]. In the operator form it is written as follows

$$\begin{aligned} &P(p_1,\ ...,p_{n1},\ k_1,\ ...,\ k_{n2},\ D)\sigma = \\ &= Q(p_1,\ ...,p_{n1},\ ...,\ k_1,\ ...,\ k_{n2},\ D)\varepsilon \end{aligned} \tag{1}$$

where $D=\dfrac{d}{dt}$, $P(p_1,\ ...,p_{n1},\ k_1,\ ...,\ k_{n2},\ S)$ and $Q(p_1,\ ...,p_{n1},\ k_1,\ ...,\ k_{n2},\ S)$ are polynomials (from S), which coefficients are dependent on the parameters of the component elasticity (p_i) and viscosity (k_i), n_1 and n_2 - the number of elasticity and viscosity elements included into the block diagram model.

Then we accept such definitions.

Definition 1. Two one-dimensional models are equivalent if the following conditions hold

$$\frac{Q_1(S)}{P_1(S)} \equiv \frac{Q_2(S)}{P_2(S)}; \tag{2}$$

$$\deg P_1 = \deg P_2, \tag{3}$$

where P_1, Q_1 и P_2, Q_2 are equation polynomials of the first and the second models.

Definition 2.

The model is called multiple, if there is an equivalent model with a less number of elements and non-multiple otherwise. It is easy to ascertain that concerning the Kelvin and Maxwell models the following facts are true.

Statement 1.

Condition $\dfrac{p_i^K}{k_i^K} \neq \dfrac{p_j^K}{k_j^K}$ $(\dfrac{p_i^M}{k_i^M} \neq \dfrac{p_j^M}{k_j^M});i \neq j$ is necessary and sufficient for non-multiplicity of models shown in Fig. 1 (Fig. 8) with $i,\ j=\overline{1,n}$, and for models – in fig. 2 – 4 (Fig. 5-7) with $i,\ j=\overline{2,n}$.

Statement 2. Any multiple Kelvin or Maxwell model is equivalent to a non-multiple model of the same type, i.e. shown in the same figure, but with less n.

The equations of non-multiple Kelvin and Maxwell models are presented in Table 1 (see Appendix).

Now we formulate the main result for the one-dimensional case.

Table 1 Non-multiple Kelvin and Maxwell model equations

Figure number	
1	$[\sum\limits_{j=1}^{n}\frac{1}{k_j^K}\cdot\mathop{\Pi}\limits_{i\neq j}(\frac{p_i^K}{k_i^K}+D)]\,\sigma=[\mathop{\Pi}\limits_{i=1}^{n}(\frac{p_i^K}{k_i^K}+D)]\,\varepsilon$
2	$[\frac{1}{p_1^K}\mathop{\Pi}\limits_{i=2}^{n}(\frac{p_i^K}{k_i^K}+D)+\sum\limits_{j=2}^{n}\frac{1}{k_j^K}\cdot\mathop{\Pi}\limits_{i\neq 1,j}(\frac{p_i^K}{k_i^K}+D)]\,\sigma=[\mathop{\Pi}\limits_{i=2}^{n}(\frac{p_i^K}{k_i^K}+D)]\,\varepsilon$
3	$[\frac{1}{k_1^K}\mathop{\Pi}\limits_{i=2}^{n}(\frac{p_i^K}{k_i^K}+D)+D\sum\limits_{j=2}^{n}\frac{1}{k_j^K}\cdot\mathop{\Pi}\limits_{i\neq 1,j}(\frac{p_i^K}{k_i^K}+D)]\,\sigma=[D\mathop{\Pi}\limits_{i=2}^{n}(\frac{p_i^K}{k_i^K}+D)]\,\varepsilon$
4	$[\frac{1}{P_1^K}\mathop{\Pi}\limits_{i=1}^{n}(\frac{p_i^K}{k_i^K}+D)+D\sum\limits_{j=2}^{n}\frac{1}{k_j^K}\cdot\mathop{\Pi}\limits_{i\neq 1,j}(\frac{p_i^K}{k_i^K}+D)]\,\sigma=[D\mathop{\Pi}\limits_{i=2}^{n}(\frac{p_i^K}{k_i^K}+D)]\,\varepsilon$
5	$[\mathop{\Pi}\limits_{i=1}^{n}(\frac{p_i^M}{k_i^K}+D)]\,\sigma=[D\sum\limits_{j=1}^{n}p_j^M\cdot\mathop{\Pi}\limits_{i\neq 1,j}(\frac{p_i^M}{k_i^M}+D)]\,\varepsilon$
6	$[\mathop{\Pi}\limits_{i=2}^{n}(\frac{p_i^M}{k_i^M}+D)]\,\sigma=[P_1^M\mathop{\Pi}\limits_{i=2}^{n}(\frac{p_i^M}{k_i^M}+D)+D\sum\limits_{j=2}^{n}P_j^m\cdot\mathop{\Pi}\limits_{i\neq 1,j}(\frac{p_i^M}{k_i^M}+D)]\,\varepsilon$

The end of the Table 1

Figure number	
7	$[\Pi_{i=2}^{n}(\frac{p_i^M}{k_i^M}+D)]\,\sigma=[k_1^M\,\Pi_{i=2}^{n}(\frac{p_i^M}{k_i^M}+D)+D\sum_{j=2}^{n}p_j^M\cdot\Pi_{i\neq1,j}(\frac{p_i^M}{k_i^M}+D)]\,\varepsilon$
8	$[\Pi_{i=2}^{n}(\frac{p_i^M}{k_i^M}+D)]\,\sigma=[k_1^M\,\Pi_{i=1}^{n}(\frac{p_i^M}{k_i^M}+D)+\sum_{j=2}^{n}P_j^m\cdot\Pi_{i\neq1,j}(\frac{p_i^M}{k_i^M}+D)]\,\varepsilon$

Theorem 1

For any Kelvin model there is an equivalent Maxwell model and, conversely, for any Maxwell model there is an equivalent Kelvin model.

First we note that proving this theorem according to Statement 2 we can restrict ourselves only by non-multiple models, and considering the form of the model equations presented in Table. 1, the following inequalities are supposed to be fulfilled

$$\frac{p_l^Y}{k_l^Y}<\ldots<\frac{p_n^Y}{k_n^Y},$$

where $Y=K,\,M$, $l=1$ – for models in Fig. 1,8; $l=2$ for models in Fig. 2-7. Now, for these models, it is sufficient to show equations (we will call them transfer equation), which allow us to find an equivalent model parameter values by the parameter values of the model. These equations are shown below in Tables 2, 3 (see Appendix).

Table 2 Transfer equations from Maxwell models to Kelvin models

Figure number	Transfer equations
1.1 and 1.5	$k_j^K=k_1^M\cdot\frac{\Pi_{i\neq j}^{n}(\lambda_j^M-\lambda_i^M)}{\Pi_{i\neq1}^{n}(\lambda_j^M+\frac{p_i^M}{k_i^M})}$; $p_j^k=-\lambda_j^M k_1^M\cdot\frac{\Pi_{i\neq j}^{n}(\lambda_j^M-\lambda_i^M)}{\Pi_{i\neq1,j}^{n}(\lambda_j^M+\frac{p_i^M}{k_i^M})}$; $j=\overline{1,n}$ $\lambda_1^M>\ldots>\lambda_n^M$ - roots of the equation $\sum_{i=2}^{n}\frac{p_i^M k_i^M S}{p_i^M+k_i^M S}+p_1^M+k_1^M S=0$
1.2 and 1.6	$p_1^K=\frac{1}{\sum_{j=1}^{n}p_j^M}$; $k_j^K=\frac{1}{\sum_{j=1}^{n}p_j^M}\cdot\frac{\Pi_{i\neq1,j}^{n}(\lambda_j^M-\lambda_i^M)}{\Pi_{i=2}^{n}(\lambda_j^M+\frac{p_i^M}{k_i^M})}$; $p_j^K=-\lambda_j^M k_j^M$; $j=\overline{2,n}$ $\lambda_1^M>\ldots>\lambda_n^M$ – roots of the equation $\sum_{i=2}^{n}\frac{p_i^M k_i^M S}{p_i^M+k_i^M S}+p_1^M=0$
1.3 and 1.7	$k_1^K=\sum_{i=1}^{n}k_i^M$; $k_j^k=k_1^M\cdot\lambda_j^M\cdot\frac{\Pi_{i\neq1,j}^{n}(\lambda_j^M-\lambda_i^M)}{\Pi_{i=2}^{n}(\lambda_j^M+\frac{p_i^M}{k_i^M})}$; $p_j^K=-\lambda_j^M k_j^M$; $j=\overline{2,n}$ $\lambda_2^M>\ldots>\lambda_n^M$ – roots of the equation $\sum_{i=2}^{n}\frac{p_i^M k_i^M}{p_i^M+k_i^M S}+k_1^M=0$
1.4 and 1.8	$p_1^K=\sum_{1}^{n}p_j^M$; $k_1^K=\sum_{1}^{n}k_j^M$; $k_j^K=p_1^K\cdot\frac{\lambda_j^M\,\Pi_{i\neq1,j}^{n}(\lambda_j^M-\lambda_i^M)}{\Pi_{i=1}^{n}(\lambda_j^M+\frac{p_i^M}{k_i^M})}$; $p_j^k=\frac{1}{\sum_{i=1}^{n}p_i^M}\cdot\frac{\Pi_{i=1}^{n}(\lambda_j^M+\frac{p_1^M}{k_1^M})}{\lambda_j^M\,\Pi_{i\neq1,j}^{n}(\lambda_j^M-\lambda_i^M)}$; $p_j^K=-\lambda_j^M k_j^K$ $\lambda_2^M>\ldots>\lambda_n^M$ – roots of the equation $\sum_{i=1}^{n}\frac{p_i^M k_i^M}{p_i^M+k_i^M S}=0$

Table 3 Transfer equations from Kelvin models to Maxwell models

Figure number	Transfer equations
1.5 and 1.1	$k_1^M = \dfrac{1}{\sum_{j=1}^{n} \dfrac{1}{k_j^K}}$; $p_1^M = \dfrac{1}{\sum_{j=1}^{n} \dfrac{1}{p_j^K}}$; $p_j^M = k_1^M \cdot \dfrac{\mathop{\Pi}\limits_{i=1}^{n} (\lambda_j^K + \dfrac{p_i^K}{k_i^k})}{\lambda_j^K \mathop{\Pi}\limits_{i \neq 1,j} (\lambda_j^K - \lambda_i^K)}$; $j = \overline{2,n}$; $k_j^M = -\dfrac{1}{\lambda_j^K} \cdot p_j^M$ $\lambda_2^K > ... > \lambda_n^K$ - roots of the equation $\sum_{i=1}^{n} \dfrac{1}{p_i^K + k_i^K S} = 0$
1.6 and 1.2	$p_1^M = \dfrac{1}{\sum_{i=1}^{n} \dfrac{1}{p_i^K}}$; $p_j^M = p_1^K \cdot \dfrac{\mathop{\Pi}\limits_{i=2}^{n} (\lambda_j^K + \dfrac{p_i^K}{k_i^k})}{\lambda_j^k \mathop{\Pi}\limits_{i \neq 1,j} (\lambda_j^K - \lambda_i^K)}$ $k_j^M = -\dfrac{1}{\lambda_j^K} \cdot p_j^M$ $\lambda_2^K > ... > \lambda_n^K$ - roots of the equation $\sum_{i=2}^{n} \dfrac{1}{p_i^K + k_i^K S} + \dfrac{1}{p_1^K} = 0$
1.7 and 1.3	$k_1^M = \dfrac{1}{\sum_{i=1}^{n} \dfrac{1}{k_i^K}}$; $k_j^M = -\dfrac{1}{\lambda_j^K} \cdot \dfrac{1}{\sum_{j=1}^{n} \dfrac{1}{k_i^K}} \cdot \dfrac{\mathop{\Pi}\limits_{i=2}^{n} (\lambda_j^K + \dfrac{p_i^K}{k_i^K})}{\mathop{\Pi}\limits_{i \neq 1,j} (\lambda_j^K - \lambda_i^K)}$; $p_j^M = \dfrac{1}{\sum_{j=1}^{n} \dfrac{1}{k_i^K}} \cdot \dfrac{\mathop{\Pi}\limits_{i=2}^{n} (\lambda_j^K + \dfrac{p_i^K}{k_i^k})}{\mathop{\Pi}\limits_{i \neq 1,j} (\lambda_j^K - \lambda_i^K)}$ $j = \overline{2,n}$ $\lambda_2^K > ... > \lambda_n^K$ - roots of the equation $\sum_{i=2}^{n} \dfrac{1}{p_i^K + k_i^K S} + \dfrac{1}{k_i^K S} = 0$
1.8 and 1.4	$p_j^M = p_1^K \cdot \dfrac{\mathop{\Pi}\limits_{i=2}^{n} (\lambda_j^K + \dfrac{p_i^K}{k_i^K})}{\mathop{\Pi}\limits_{i \neq j} (\lambda_j^K - \lambda_i^K)}$; $k_j^M = -\dfrac{1}{\lambda_j^K} \cdot p_j^M$; $j = \overline{1,n}$ $\lambda_2^K > ... > \lambda_n^K$ - roots of the equation $\sum_{i=2}^{n} \dfrac{1}{p_i^K + k_i^K S} + \dfrac{1}{p_1^K} + \dfrac{1}{k_1^K S} = 0$

Three-dimensional models

When using a 3-D models of linear isotropic material, built on the basis of the mechanical block diagrams, each elasticity element is characterized by a modulus of elasticity in shift (G) and the Poisson's ratio (μ^e), and each viscosity element is characterized by the viscous resistance coefficient (η) and strain coefficient (μ^V). The relations connecting the stress deviator components ($\overline{\sigma_{ij}}$), strain ones ($\overline{\varepsilon_{ij}}$) and also mean stress ($\sigma_c$) and strain ($\varepsilon_c$) can be written in the form of differential equations, which are defined as follows.

Let $P(p_1, \dots, p_{n1}, k_1, \dots, k_{n2}, D)$ and $Q(p_1, \dots, p_{n1}, k_1, \dots, k_{n2}, D)$ are operators of the one-dimensional model equations (1), and

$$P_g(D) = P(2G_1, \dots, 2G_{n1}, 2\eta_1, \dots, 2\eta_{n2}, D); \quad (4)$$

$$Q_g(D) = P(2G_1, \dots, 2G_{n1}, 2\eta_1, \dots, 2\eta_{n2}, D); \quad (5)$$

$$P_c(D) = P(p_1^*, \dots, p_{n1}^*, k_1^*, \dots, k_{n2}^*, D); \quad (6)$$

$$Q_c(D) = Q(p_1^*, \dots, p_{n1}^*, k_1^*, \dots, k_{n2}^*, D); \quad (7)$$

where $p_i^* = \dfrac{2G_i(1+\mu_i^e)}{1-2\mu_i^e}; \quad i = \overline{1,\ n_1}$,

$$k_i^* = \frac{2\eta_i(1+\mu_i^V)}{1-2\mu_i^V}; \quad i = \overline{1,\ n_2}.$$

Then

$$P_g(D)\overline{\sigma_{ij}} = Q_g(D)\overline{\varepsilon_{ij}},\ i, j = \overline{1,3}, \quad (8)$$

$$P_c(D)\sigma_c = Q_c(D)\varepsilon_c. \quad (9)$$

Definition 3. Two three-dimensional models are equivalent, if the following conditions hold

$$\frac{Q_{1g}(S)}{P_{1g}(S)} \equiv \frac{Q_{2g}(S)}{P_{2g}(S)};\ \frac{Q_{1c}(S)}{P_{1c}(S)} \equiv \frac{Q_{2c}(S)}{P_{2c}(S)}; \quad (10)$$

$$\deg\ P_{1g} = \deg\ P_{2g};\ \deg\ P_{1c} = \deg\ P_{2c}. \quad (11)$$

In a three- dimensional case theorem 1 is false, i.e. there is a Kelvin (Maxwell) model to which there is no equivalent Maxwell (Kelvin) model. However, one can select a wide class of models to which the statement of this theorem is true.
Theorem 2.

If these conditions hold for a Kelvin (Maxwell) model

$$\frac{G_i^K}{\eta_i^K} \ne \frac{G_j^K}{\eta_j^K} \qquad \left(\frac{G_i^M}{\eta_i^M} \ne \frac{G_j^M}{\eta_j^M}\right); \quad (12)$$

$$\frac{1+\mu_i^{e,K}}{1-2\mu_i^{e,K}} \cdot \frac{1-2\mu_i^{V,K}}{1+\mu_i^{V,K}} \cdot \frac{G_i^K}{\eta_i^k} \ne \frac{1+\mu_j^{e,K}}{1-2\mu_j^{e,K}} \times \frac{1-2\mu_j^{V,K}}{1+\mu_j^{V,K}} \cdot \frac{G_j^K}{\eta_{j,k}^K}, \quad (13)$$

$$\left(\frac{1+\mu_i^{e,M}}{1-2\mu_i^{e,M}} \cdot \frac{1-2\mu_i^{V,M}}{1+\mu_i^{V,M}} \cdot \frac{G_i^k}{\eta_i^k} \ne \frac{1+\mu_j^{e,M}}{1-2\mu_j^{e,M}} \times \frac{1-2\mu_j^{V,M}}{1+\mu_j^{V,M}} \cdot \frac{G_j^M}{\eta_{j,k}^M}\right), \quad (14)$$

with $i \ne j$; $i, j = \overline{1,n}$ for Fig. 1, (Fig. 8) and $i, j = \overline{2,n}$ for Fig. 2–4, (5–7), then there is an equivalent Maxwell (Kelvin) model to it.

The proof of this theorem as well as Theorem 1 is constructive. We describe the construction of transfer equations for these models. They are obtained by using the transfer equations for one-dimensional models. For example, suppose you must find a model of Maxwell, which is equivalent to Kelvin model with parameters

$$G_1^k, \dots, G_n^{\mathrm{k}}, \mu_1^{e,k}, \dots, \mu_n^{e,k}, \eta_1^k, \dots, \eta_n^k, \mu_1^{V,k}, \dots, \mu_n^{v,k}$$

We write the transfer equations for a one-dimensional model having the same block diagram as

$$\begin{aligned} p_i^M &= g_i(p_1^k, \dots, p_n^k, \dots, k_1^k, \dots, k_n^k) \\ k_i^M &= f_i(p_i^k, \dots, p_n^k, \dots, k_1^k, \dots, k_n^k) \end{aligned} \quad i = \overline{1,n}. \quad (15)$$

Then, considering that (13). (15) hold, we obtain

$$\begin{aligned} G_i^M &= \frac{1}{2} g_i(2G_1^k, \dots, 2\ G_n^{\mathrm{k}}, 2\ \eta_1^k, \dots, \eta_n^k) \\ \eta_i^M &= \frac{1}{2} f_i(2G_1^k, \dots, 2G_n^k, .2\eta_1^{\mathrm{k}}, \dots, 2\eta_{\mathrm{n}}^{\mathrm{k}}) \end{aligned} \quad (16)$$

$$\mu_i^{e,M} = \frac{1}{2} \frac{g_i(p_1^{*,K}, \dots, p_n^{*,M}, k_1^{*,K}, \dots, k_n^{*,K}) - 2G_i^K}{g_i(p_1^{*,K}, \dots, p_n^{*,M}, k_1^{*,K}, \dots, k_n^{*,K}) + G_i^M}$$

$$\mu_i^{V,M} = \frac{1}{2} \frac{f_i(p_1^{*,K}, \dots, p_n^{*,M}, k_1^{*,K}, \dots, k_n^{*,K}) - 2\eta_i^M}{f_i(p_1^{*,K}, \dots, p_n^{*,M}, k_1^{*,K}, \dots, k_n^{*,K}) + \mu_i^M}$$

$$p_i^{*,K} = \frac{2G_i^K(1+\mu_i^{e,K})}{1-2\mu_i^{e,K}}\ k_i^{*,K} = \frac{2\eta_i^K(1+\mu_i^{V,K})}{1-2\mu_i^{V,K}}.$$

Conclusions

Thus, for multiple Kelvin and Maxwell models we have established their equivalences in a one-dimensional case and sufficient conditions have been presented for its validity in a three-dimensional case. The transfer equations from one type model to another have been found in an explicit form.

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