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The manual contains a short course of lectures on theoretical mechanics – the science of motion and interaction of absolutely solid bodies. The theoretical mechanics course is mandatory for all engineering majors. The manual was produced with a focus on teaching in a higher educational institution. This study guide includes the complete course in a condensed form. All basic concepts, formulas and theorems are given in three sections: statics, kinematics, dynamics. The material submission form is intended for accelerated study or repetition of the material.

Посібник містить короткий курс лекцій з теоретичної механіки – науки про рух і взаємодію абсолютно твердих тіл. Курс теоретичної механіки є обов’язковим для всіх інженерних спеціальностей. Посібник виготовлено для запровадження в освітній процес у закладах вищої освіти. Цей навчальний посібник містить повний курс у стислому вигляді. Усі основні поняття, формули та теореми наведено в трьох розділах: статика, кінематика, динаміка. Форма подачі матеріалу призначена для прискореного вивчення або повторення матеріалу.

Іл. 34. Табл. 1. Бібліогр. 13 найм.

INTRODUCTION

Mechanics studies the general patterns that unite the concepts of motion and interaction of bodies. In this case, bodies can be considered in three states of aggregation: solid, liquid and gaseous. In accordance with this, mechanics is divided into several scientific areas: the mechanics of an absolutely rigid body, the mechanics of a continuous medium, the mechanics of a deformable solid, and the mechanics of an ionized gas. The mechanics of an absolutely rigid body is theoretical mechanics. Continuum mechanics is a more general scientific direction that includes the mechanics of various bodies, including liquids and gases. The mechanics of a deformable solid body is usually represented by the disciplines of the theory of elasticity, the theory of plasticity, the theory of cracks. The mechanics of ionized gas is the mechanics of plasma.

Mechanics is based primarily on the study of the simplest forms of motion and the interaction of typical (simplest) material bodies. Mechanics in the first approximation abstracts from many features and actual properties of the bodies under study. This is carried out primarily by using the concepts of a material point and a system of material points. The material system can be discrete and consist of individual material points or bodies. A material system can be continuous, representing continuous distributions of matter and the physical characteristics of its state and movement. Thus, the concept of a continuous material medium or simply a continuous medium is introduced. The simplest case of a continuous medium is an unchanging medium or an absolutely rigid body. Greater abstraction introduces the concept of a variable continuous medium, which combines in mechanics both solid, elastic, plastic, liquid and gaseous bodies. In addition, in continuum mechanics, in addition to the usual bodies (metal, wood, plastic, liquid, air), special media (fields) are considered: an electromagnetic field, a radiation field, a gravitational field, and others. It must be emphasized that mechanics is based only on the most elementary physical properties of matter. Schematizing physical phenomena, mechanics does not consider the molecular structure of matter and intermolecular interactions.

Theoretical mechanics is a science that studies the mechanical movement and interaction of material bodies, i.e. change in their position relative to each other over time. Since the state of rest is a special case of mechanical motion, the task of theoretical mechanics also includes the study of the equilibrium of material bodies.

Theoretical mechanics is a science that is designed to solve a wide class of problems in continuum mechanics using the simplest methods. Due to this, theoretical mechanics now covers a large number of vital modern problems. Theoretical mechanics has a significant role in the development of world science. The work of mechanics is invested in calculating the trajectories of spacecraft, in navigational instruments of varying complexity and purpose, in the design of engines of various types, rocket technology, in solving pressing problems of the national economy, in promising tasks for the exploration of near and far space. The work of mechanics is invested in a variety of devices, structures, mechanisms, machines, without which the existence of our society and its further development is unthinkable. Construction and transportation have a long association with mechanics. All modern production processes are connected in one way or another with mechanics. Without the achievements of theoretical mechanics, it is impossible to create automated and production lines, automatic control systems for production processes, and robotics. Mechanics is used by medicine in diagnosing diseases and creating artificial organs. No modern navigation system can do without the application of the results of theoretical mechanics. Advances in the study of oscillatory systems allow them to be used to accelerate, improve and create the latest technological processes. The mechanics of the experiment is increasingly using all the possibilities that modern technology presents. It is increasingly enriched with new theories that make it possible to predict and control the course of various processes by calculation. Theoretical mechanics is constantly faced with new vitally important tasks, and there are also many important questions that have not yet been adequately investigated. The powerful development of modern mechanics is a convincing proof of the viability of classical theoretical mechanics, the fruitfulness of its links with modern science and technology. Theoretical mechanics is the basis and foundation of a large number of theoretical and applied disciplines, which in particular include: Theory of motion stability, Theory of oscillations, Dynamics of nonholonomic systems, Theory of optimal control systems, Mechanics of gyroscopic and navigation systems, Space flight mechanics, Celestial mechanics, Theory of mechanisms and machines, Creation of automated and robotic systems. Traditionally, mechanics is divided into three sections: statics, kinematics, dynamics. This classic sequence outlines the course of theoretical mechanics in this manual.

1 STATICS

1.1 Axioms of statics

Statics is a section of theoretical mechanics, in which the problems of the balance of systems of forces are considered.

Force is a measure of the mechanical interaction of bodies. Force is a vector quantity, characterized by three elements: numerical value (modulo), direction and application point. Unit of measurement - newton, kg*m/c^2, 1kN (kilonewton) = 10^3 N.

The straight line along which the force is directed, called - line of force.

Axioms (laws) of statics:

1) axiom of inertia: Under the action of mutually balancing forces, a material point (body) is at rest or moves in a straight line and uniformly.

2) the axiom of the balance of two forces: Two forces applied to an absolutely rigid body will be balanced if and only if they are equal in absolute value, act in one straight line and are directed in opposite directions.

3) axiom of addition and exclusion of balancing forces: The action of the system of forces on absolutely rigid body will not change if a balanced system of forces is added to or subtracted from it. Consequence: The action of force on absolutely solid body will not change if the point of application of the force is moved along its line of action. Those. force applied to absolutely solid body is a sliding vector.

4) the axiom of the parallelogram of forces: The resultant of two intersecting forces is applied at the point of their intersection and is represented by the diagonal of the parallelogram built on these forces.

$$\vec{R} = \vec{F}_1 + \vec{F}_2; \qquad R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$

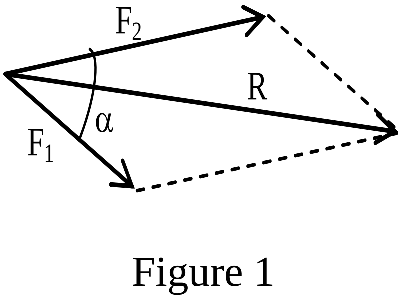


Figure 1

5) axiom of equality of action and reaction (Newton's 3rd law): Every action corresponds to an equal and oppositely directed reaction.

6) the principle of solidification: The balance of forces applied to a non-solid body is not disturbed when it solidifies.

A body is called free if its movements are not limited by anything. A body whose movement is limited by other bodies, called. not free. Bodies that limit the movement of a given body, called. connections. The forces with which the bonds act on a given body, called. bond reactions. The principle of release: Any non-free body can be considered as free if the action of the bonds is replaced by their reactions applied to the body.

The main types of bonds:

a) support on an ideally smooth surface - the reaction of the surface is directed along the normal to it, i.e. perpendicular to the tangent - normal reaction;

b) one of the contacting surfaces is a point (angle), the reaction is directed along the normal to the other surface;

c) thread - the reaction is directed along the thread to the suspension point;

d) cylindrical hinge (hinge-fixed support) - the reaction can have any direction in the plane. When solving problems, it is replaced by two mutually perpendicular components;

e) cylindrical articulated-movable support (hinge on rollers) - the reaction is directed perpendicular to the reference plane;

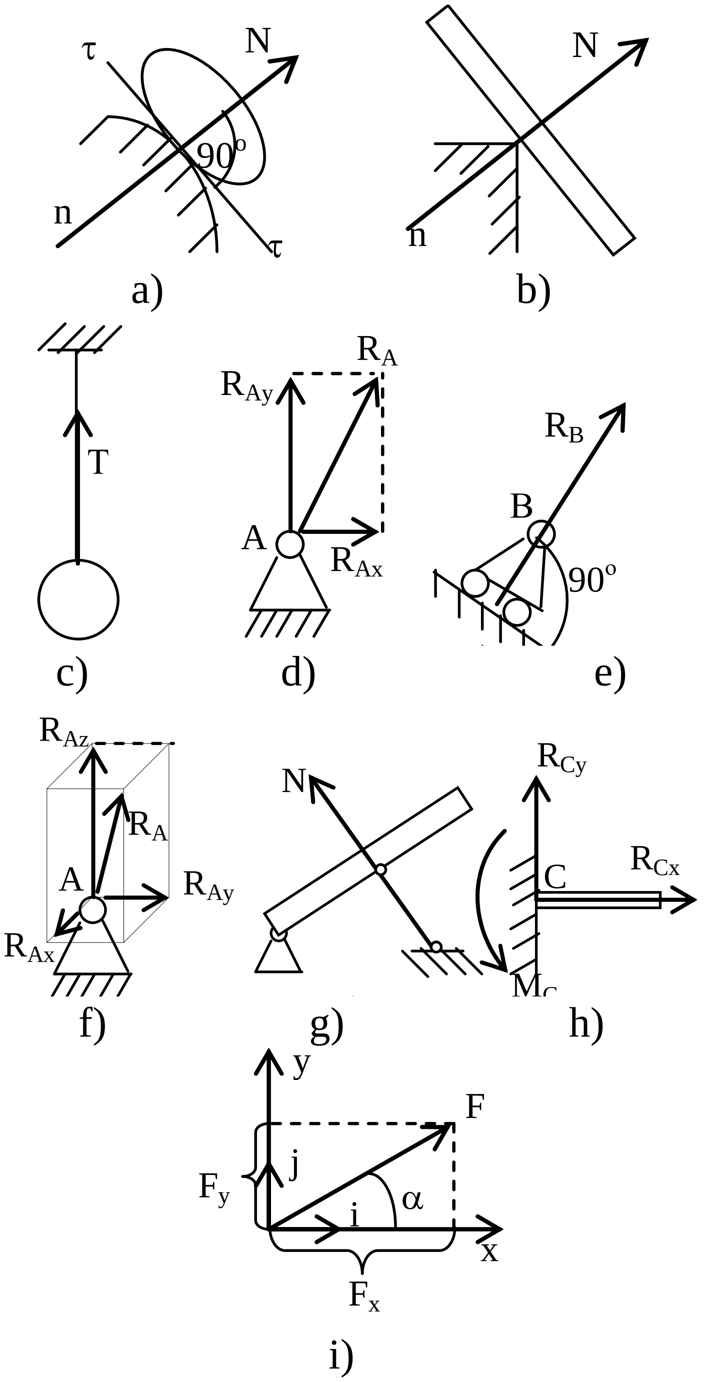
f) spherical (ball) hinge - the reaction can have any direction in space. When solving problems, it is replaced by three mutually perpendicular components;

g) a weightless rod (necessarily weightless) - the reaction is directed along the rod;

h) "blind" embedment (embedded beam) - an arbitrarily directed reaction occurs - a force and a reactive moment, also unknown in direction. The reaction is decomposed into two components.

1.2 Converging force system

System of converging forces (fig.2, i). Converging forces are called forces whose lines of action intersect at one point. The resultant of the converging forces is equal to the geometric sum of these forces and is applied at the point of their intersection. The resultant can be found geometrically. way - by building a power (vector) polygon or analytical. way by projecting forces on the coordinate axes. Force projections on the coordinate axes (for a flat system): $F_x = F\cos\alpha$; $F_y = F\cos(90-\alpha) = F\sin\alpha$; projection>0, if the direction of the force component coincides with the direction. axes.



a) b)

c) d) e)

f) g) h)

i)

Figure 2

Power module:

Force modulus:

$$F = \sqrt{F_x^2 + F_y^2};$$

direction cosines:

$$\cos\alpha = \frac{F_x}{F}; \quad \cos\beta = \frac{F_y}{F};$$

expansion of the force into components:

$$\vec{F} = F_x \cdot \vec{i} + F_y \cdot \vec{j},$$

where $\vec{i}, \vec{j}$ is the unit vector (unit vector) of the corresponding axis.

For a spatial system:

$\vec{F} = F_x \cdot \vec{i} + F_y \cdot \vec{j} + F_z \cdot \vec{k}$, $F_x = F\cos\alpha$; $F_y = F\cos\beta$; $F_z = F\cos\gamma$;

$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$; $\cos\alpha = \frac{F_x}{F}; \cos\beta = \frac{F_y}{F}; \cos\gamma = \frac{F_z}{F}$.

The projections of the resultant system of converging forces on the coordinate axes are equal to the algebraic sums of the projections of these forces on the corresponding axes:

$$R_x = \sum F_{ix}; \ R_y = \sum F_{iy}; \ R_z = \sum F_{iz}; \ R = \sqrt{R_x^2 + R_y^2 + R_z^2}.$$

Equilibrium conditions for the system. converging forces: geometric:

$$\sum \vec{F}_i = 0$$

analytical:

$$\sum F_{ix} = 0; \ \sum F_{iy} = 0; \ \sum F_{iz} = 0.$$

Theorem of three non-parallel forces: If under the action of three forces the body is in equilibrium and the lines of action of two forces intersect, then all forces lie in the same plane and their lines of action intersect at one point.

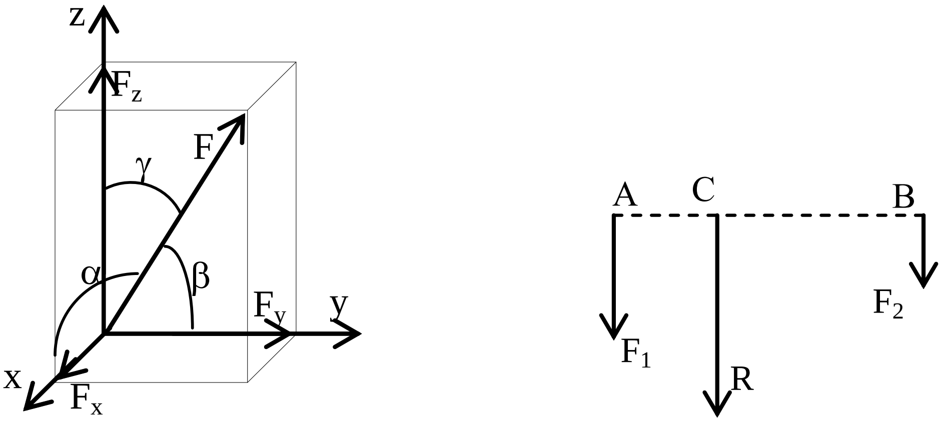


Figure 3

1.3 Pair Theory

Theory of pairs of forces. Addition of two parallel forces: the resultant of two parallel forces F1 and F2 of the same direction has the same direction, its module is equal to the sum of the modules of the forces, and the point of application divides the segment between the points of application of forces into parts inversely proportional to the force modules: $R = F_1 + F_2$; $AC/BC = F_2/F_1$. The resultant of two oppositely directed parallel forces has a direction of force greater in absolute value and a module equal to the difference in the force modules.

The system of two parallel forces, equal in magnitude and directed in different directions, called. a couple of forces. The shortest distance between the lines of action of these forces is called. shoulder pair "h". The action of a pair of forces is characterized by its moment. The moment of a pair of forces $M = F \cdot h$ is the product of the modulus of one of the forces of the pair and its arm.

The moment of a pair of forces is a vector directed perpendicular to the plane of forces, so that, if we look towards it, we see the rotation of the pair against the clockwise direction. M>0, if counterclockwise, M<0 - clockwise (in fig.4, a) M>0).

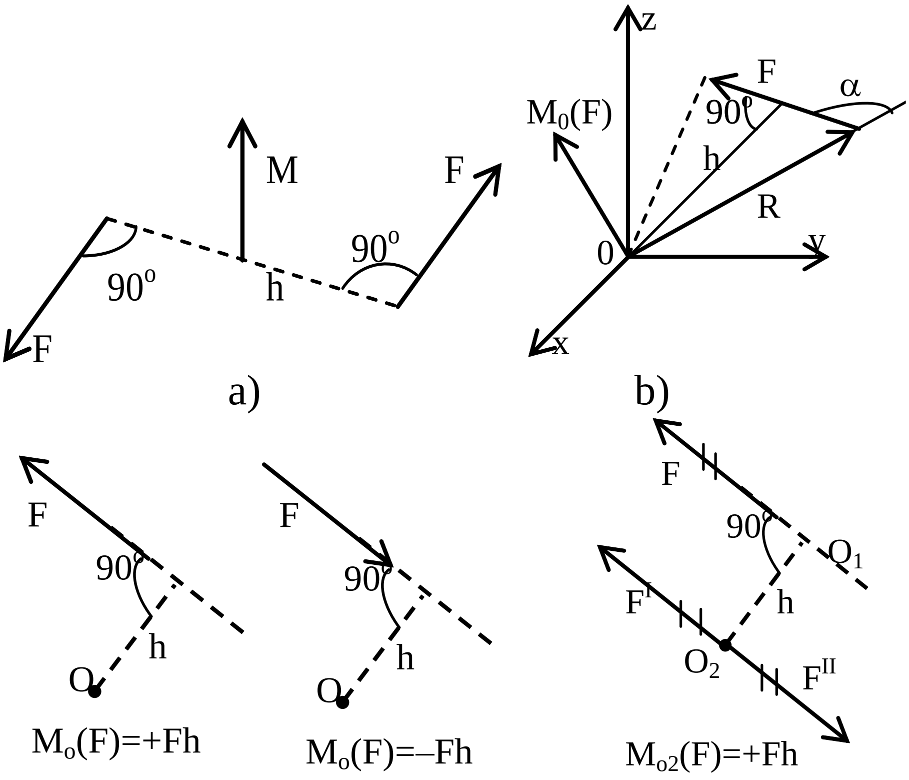
Pair theorems.

1) Two pairs lying in the same plane can be replaced by one pair lying in the same plane, with a moment equal to the sum of the moments of these two pairs. $\vec{M} = \vec{M}_1 + \vec{M}_2$.

2) Two pairs having geometrically equal moments are equivalent.

3) Without violating the state of a rigid body, a pair of forces can be transferred in the plane of its action. Those. the moment of a pair of forces is a free vector.

4) The system of several pairs of forces is equivalent to one pair, the moment of which is equal to the vector sum of the moments of these pairs. Those the system of pairs is reduced to one pair, the moment of which is equal to the sum of the moments of all pairs. The condition for the equilibrium of pairs of forces: $\sum \vec{M}_i = 0$ - the geometric sum of their moments is 0. Pairs of forces located in the same plane are mutually balanced if the algebraic sum of their moments $\sum M_i=0$.



a) b)

c) d)

Figure 4

The moment of force relative to a point is a vector numerically equal to the product of the modulus of force and the arm and directed perpendicular to the plane containing the force and the point in such a direction that, looking towards it, you can see the force tending to turn counterclockwise. Shoulder "h" - the shortest distance from the point to the line of action of the force.

$$\vec{M}_0(\vec{F}) = \vec{R} \times \vec{F}$$

- the moment of force is equal to the vector product of the vector by the vector. Vector product modulus:

$$M_0(\vec{F}) = R \cdot F \cdot \sin\alpha = F \cdot h.$$

For flat system forces usually find not the moment vector, but only its modulus:

$$M_0(\vec{F}) = \pm F \cdot h,$$

>0 - counterclockwise; <0 - hourly page.

Properties of the moment of force:

1) the moment of force does not change when the point of application of force is transferred along its line of action;

2) the moment of force relates. points = 0 only when the force = 0 or when the line of action of the force passes through the point (i.e. arm = 0).

If x,y,z are the coordinates of the force application point, Fx, Fy, Fz are the projections of the force on the coordinate axes and point 0 is the origin of coordinates, then

$$\vec{M}_0(\vec{F}) = \vec{R} \times \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = (yF_z - zF_y)\,\bar{i} + (zF_x - xF_z)\,\bar{j} + (xF_y - yF_x)\,\bar{k},$$

$$M_{0x}(\vec{F}) = yF_z - zF_y; \qquad M_{0y}(\vec{F}) = zF_x - xF_z; \quad M_{0z}(\vec{F}) = xF_y - yF_x.$$

The main vector is the vector sum of all forces applied to the body. The main moment relative to the center is the vector sum of the moments of all forces applied to the body relative to the same center.

Theorem (lemma) on the parallel transfer of force: the force applied at any point is solid. body, equivalent to the same force applied at any other point of this body, and a pair of forces, the moment of which is equal to the moment of this force relative to the new point of application.

1.4 Flat force system

A flat system of forces is a system of forces located in the same plane. The system of forces is reduced to one force - the main vector and to a pair of forces, the moment of which is equal to the main moment. The moment of a pair of forces is directed perpendicular to the plane in which the forces lie. In planar systems, there is no need to use the vector representation of the moment.

Varignon's theorem - if a flat system of forces is reduced to a resultant, then its moment relative to any point is equal to the algebraic (i.e., taking into account the sign) sum of the moments of all forces relates the same point.

Equilibrium conditions squre systems forces: vector:

$$\sum \vec{F}_k = 0; \ \sum M_O(\vec{F}_k) = 0;$$

analytical:

$$\sum F_{kx} = 0; \ \sum F_{ky} = 0; \ \sum M_O(\vec{F}_k) = 0 \text{ or}$$

$$\sum M_A(\vec{F}_k) = 0; \sum M_B(\vec{F}_k) = 0; \sum M_C(\vec{F}_k) = 0$$

where A, B, C are points that do not lie on one straight line, or

$$\sum M_A(\vec{F}_k) = 0; \sum M_B(\vec{F}_k) = 0; \sum F_{kx} = 0,$$

the "x" axis is not perpendicular to the segment AB.

1.5 Friction forces

Equilibrium of bodies in the presence of friction. Coulomb's law (Amont-Coulomb's law): the maximum cohesive force is proportional to the normal pressure of the body on the plane

$$F^{\max} = fac \cdot N,$$

f_{ac} – adhesion coefficient (depends on the material, the state of the surfaces, is determined experimentally). The direction of the adhesion force is opposite to the direction of the movement that would occur in the absence of adhesion. When a body slides over a rough surface, a sliding friction force is applied to it. Its direction is opposite to the speed of the body

$$F = f \cdot N,$$

f is the coefficient of sliding friction (determined empirically). $f < f_{ac}$. The reaction of a rough (real) surface, in contrast to a perfectly smooth one, has two components: a normal reaction and an adhesion force (or a friction force during movement). Angle $\varphi_{\text{сц}}$ –coupling angle ($\varphi_{\text{тр}}$ – angle of friction)

$$\mathrm{tg}\varphi_c = f_{ac}\,; \quad (\mathrm{tg}\varphi = f).$$

A cone with a vertex at the point of contact of the bodies, the generatrix of which forms the angle of adhesion (angle of friction) with the normal to the surfaces of the body. clutch cone (friction cone). In order for the body to start moving, it is necessary (and sufficient) that the resultant of the active forces be outside the cone of friction. Rolling friction is the resistance that occurs when one body rolls on the surface of another. The reason for its appearance is in the deformation of the rink and the plane at the point of their contact and the displacement of the normal reaction in the direction of possible movement. $M = f_k N$ – rolling friction moment, f_k – rolling friction coefficient; has the dimension of length.

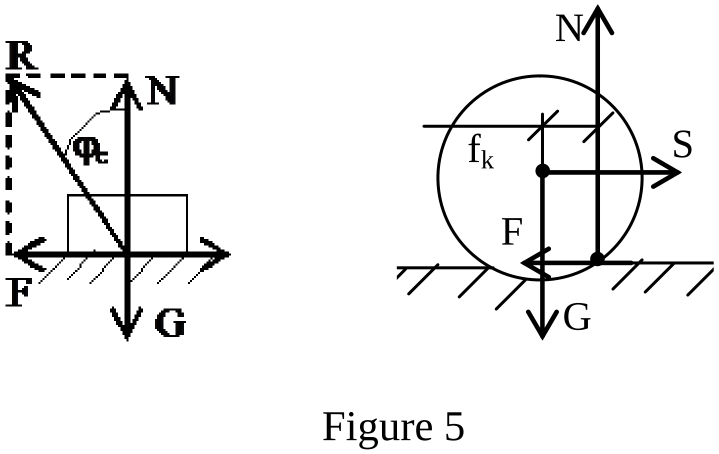


Figure 5

1.6 Spatial force system

Spatial system of forces. Moments of force about an axis is a scalar value equal to the moment of the projection of this force onto a plane, the transverse axis, taken relative to the point of intersection of the axis with the plane. Moment>0, if towards the axis, we see a rotation that realizes the direction of the counterclockwise

$$M_Z(\vec{F}) = M_Z(\vec{F}_{xy}) = M_O(\vec{F}_{xy}) = \pm F_{xy} h,$$

On fig.6 M>0. The moments of force about the axis are 0: 1) if the force is proportional to the axis ($F_{xy}=0$), 2) if the action of the scoping line is the axis (h=0); those. if the axis and belongs to the same side. Analytical expressions of the moments of force relative to the coordinate axes: $M_x(\vec{F})=yF_z - zF_y$; $M_y(\vec{F})=zF_x - xF_z$; $M_z(\vec{F})=xF_y - yF_x$..

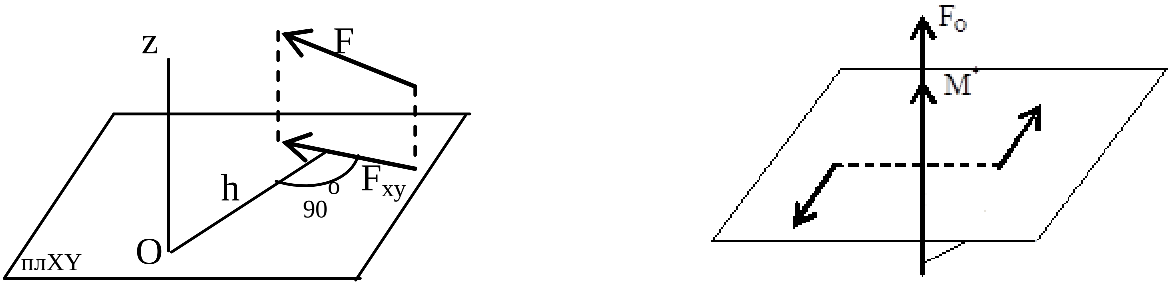


Figure 6

Bringing the spatial system of forces to a given center is solved using the theorem on the parallel transfer of force. Any system of forces acting on an absolutely rigid body, when reduced to an arbitrarily taken center O, is replaced by one force R equal to the main vector of the system and applied at the center of reduction O, and one pair with a moment MO equal to the main moment of the system relative to relative to the center O (the main vector is the vector sum of all forces applied to the body; the main moment relative to the center is the vector sum of the moments of all forces applied to the body relative to the same center). Static invariants of spaces. syst. forces - such characteristics of this system that remain unchanged when the center of reference is changed. 1st invariant – main vector (squared modulus of the main vector):

$$I_1 = F_o^2 = F_x^2 + F_y^2 + F_z^2;$$

The 2nd invariant is the scalar product of the main vector and the main moment:

$$I_2 = \vec{F}_O \cdot \vec{M}_O = F_x \cdot M_x + F_y \cdot M_y + F_z \cdot M_z.$$

When the center of reference is changed, the projection of the main moment on the direction of the main vector M * does not change.

$$M^* = \frac{\vec{M}_O \cdot \vec{F}_O}{F_O} = \frac{I_2}{\sqrt{I_1}}$$

The totality of a force and a pair of forces, with a moment located in a plane perpendicular to the line of action of this force, called. dynamo (power screw). The system is reduced to dynamism if the second static invariant is not equal to 0. The straight line along which and are directed is called the central axis of the system of forces. The central axis of the system of forces is the locus of points in space, relative to which the main moments of a given system of forces have the largest modulus Mmin=M* and are directed along this axis. If the main vector

$$\vec{F}_O = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

and the main moment

$$\vec{M}_O = M_{Ox} \vec{i} + M_{Oy} \vec{j} + M_{Oz} \vec{k},$$

then the equations of the central axis:

$$\frac{M_{Ox} - (yF_z - zF_y)}{F_x} = \frac{M_{Oy} - (zF_x - xF_z)}{F_y} = \frac{M_{Oz} - (xF_y - yF_x)}{F_z}$$

Table

Cases of reduction of the spatial system of forces

	$I_2 = \vec{F}_O \cdot \vec{M}_O$	F_0	M_0	Cast case
1	$I_2 \neq 0$	$F_0 \neq 0$	$M_0 \neq 0$	Dynama
2	$I_2 = 0$	$F_0 \neq 0$	$M_0 \neq 0$; $M_0 = 0$	Resultant
3	$I_2 = 0$	$F_0 = 0$	$M_0 \neq 0$	Power couple
4	$I_2 = 0$	$F_0 = 0$	$M_0 = 0$	0

Varignon's theorem (the theorem on the moment of the resultant force): the moment of the resultant force relative to any point = the geometric sum of the moments of the constituent forces relative to the same point.

Conditions for the equilibrium of spaces systems of forces:

$$\sum F_{kx}=0;\ \sum F_{ky}=0;\ \sum F_{kz}=0;\ \sum M_x(F_k)=0;\ \sum M_y(F_k)=0;\ \sum M_z(F_k)=0.$$

1.7 Center of gravity

Equilibrium conditions for a system of parallel forces (||z, || forces):

$$\sum F_{kz}=0;\ \sum M_x(F_k)=0;\ \sum M_y(F_k)=0.$$

The center of parallel forces is the point through which the line of action of the equal-acting system of parallel forces passes for any rotation of these forces near their points of application in the same direction and at the same angle. Coordinates of the center of parallel forces:

$$x_C = \frac{\sum F_{kx} \cdot x_k}{\sum F_{kx}}, \text{ etc.}$$

The center of gravity of a solid body is a point invariably associated with this body, through which the line of action of the resultant forces of gravity of the particles of the body passes at any position of the body in space. In this case, the gravity field is considered homogeneous, i.e. the forces of gravity of the particles of the body are parallel to each other and remain constant for any rotation of the body. Center of gravity coordinates:

$$x_C = \frac{\sum p_{kx} \cdot x_k}{P};\ y_C = \frac{\sum p_{ky} \cdot y_k}{P};\ z_C = \frac{\sum p_{kz} \cdot z_k}{P},$$

where $P=\sum p_k$, x_k, y_k, z_k are the coordinates of the points of application of gravity forces p_k. The center of gravity is a geometric point and can also lie outside the body (for example, a ring). Center of gravity of a flat figure:

$$x_C = \frac{\sum x_k \cdot \Delta F_k}{F} \quad ,$$

ΔF_k is the elementary area, F is the area of the figure. If the area cannot be divided into several finite parts, then

$$x_C = \frac{1}{F} \int\limits_{(F)} x dF .$$

If a homogeneous body has an axis of symmetry, then the center of gravity of the body is located on this axis. Center of gravity: arcs of a circle with a central angle 2α:

$$x_C = R \frac{\sin\alpha}{\alpha};$$

circular sector:

$$x_C = \frac{2}{3} R \frac{\sin\alpha}{\alpha};$$

triangle: at the point of intersection of medians (1/3 of the median from the base).

The static moment of the area of a flat figure is the sum of the products of the elementary areas that make up the area of the figure and the algebraic values of the distances to some axis.

$$S_x=\sum y_i \cdot \Delta F_i = F \cdot y_c; \quad S_y=\sum x_i \cdot \Delta F_i = F \cdot x_c.$$

Auxiliary theorems for determining the position of the center of gravity:

Theorem 1. If a homogeneous body has an axis of symmetry, then the center of gravity of the body is located on this axis.

Theorem 2. If a homogeneous body has a plane of symmetry, then its center of gravity is in this plane.

Figure 7

Theorem 3. The volume of a body of revolution obtained by rotating a flat figure around an axis that lies in the plane of the figure, but does not intersect it, is equal to the product of the figure's area and the circumference circumscribed by its center of gravity, $V=2\pi x_c F$.

Theorem 4. The area of the surface of revolution obtained by rotating a plane curve around an axis that lies in the plane of this curve, but does not intersect it, is equal to the product of the length of this curve and the circumference circumscribed by its center of gravity, $F=2\pi x_c L$.

Determining the position of the center of gravity of a flat figure with a part cut out of it, we can consider the area of this part to be negative, and then method of negative areas (volumes) :

$$x_c = \frac{F_1 x_1 - F_2 x_2}{F_1 - F_2}, \text{ etc.}$$

CONTROL QUESTIONS

1 What is static?
2 What is called a material point and an absolutely solid body?
3 What is force?
4 What elements determine the force?
5 Equalizing and balancing forces.
6 Equivalent systems of forces.
7 External and internal forces.
8 Axiom 1 (about two forces).
9 Axiom 2.
10 Corollary with axioms 1 and 2.
11 Axiom 3 (about the parallelogram of forces).
12 Axiom 4 (on the equality of action and counteraction).
13 Axiom 5 (hardening principle).
14 Axiom of elms.

15 What are elms and reactions of elms?

16 Elms and their reactions (smooth and rough support surfaces).

17 Elms and their reactions (flexible elm).

18 Elms and their reactions (hinged-movable and hinged-fixed supports).

19 Elms and their reactions (spherical joint).

20 Elms and their reactions (bearing and support) .

21 Elms and their reactions (ideal rod elm) .

22 Elms and their reactions (hard pinching).

23 Problems of statics.

24 What is a system of convergent forces?

25 Equivalent systems of convergent forces.

26 Projection of force on the axis.

27 Projection of force on a plane.

28 Analytical method of determining the equivalent.

29 Do the reactions of elms remain constant when external forces change?

30 Formulate the rules for building a power triangle and multi-corner

31 Geometric conditions of equilibrium of the spatial system of convergent forces.

32 Theorem about three non-parallel forces.

33 Geometric equilibrium condition of a plane system of convergent forces.

34 Analytical conditions of equilibrium of the spatial system of convergent forces.

35 Analytical equilibrium conditions of a plane system of convergent forces.

36 Equation of moments for a plane system of convergent forces.

37 Varignon's theorem for a system of convergent forces.

38 Geometric conditions of equilibrium of the spatial system of force pairs.

39 Analytical conditions of equilibrium of the spatial system of pairs of forces.

40 Conditions of equilibrium of a flat system of force pairs.

41 Basic equilibrium conditions of an arbitrary planar system of forces.

42 Equivalent forms of equilibrium conditions of an arbitrary planar system of forces.

43 Conditions of equilibrium of parallel forces on a plane.

44 Conditions of equilibrium of parallel forces in space.

45 Geometric conditions of equilibrium of an arbitrary spatial system of forces.

46 Analytical equilibrium conditions of an arbitrary spatial system of forces.

47 Moment of force relative to the center.

48 Moment of force relative to the center as a vector.

49 Moment of a couple of forces.

50 Moment of a pair of forces as a vector.

51 Moment of force relative to the axis.

52 Moment of force relative to coordinate axes.

53 When is the moment of force relative to the axis equal to zero?

54 How to determine the modulus and direction of the principal vector of an arbitrary spatial system of forces?

55 How to determine the modulus and direction of the principal moment of an arbitrary spatial system of forces?

56 Principal vector and principal moment of an arbitrary planar system of forces.

57 Theorem on parallel transfer of forces.

58 Addition of two parallel forces directed in one direction.

59 Addition of two parallel forces directed in opposite directions.

60 Equivalence of force pairs (Theorem 1).

61 Equivalence of pairs of forces (Theorem 2).

62 Addition of pairs of forces in space.

63 Adding pairs of forces on a plane.

64 The basic theorem of statics for an arbitrary planar system of forces.

65 The basic theorem of statics for an arbitrary spatial system of forces.

66 Distributed loads (evenly and linearly).

67 Distributed loads (along the arc of a circle).

68 Sliding friction force.

69 Friction angle and cone.

70 Moment of resistance to rolling and rotation.

71 When does the phenomenon of jamming occur?

72 What is a flat farm? Truss stiffness condition.

73 What is a flat farm? Flat farm elements.

74 Methods of truss calculations (cutting nodes).

75 Farm calculation methods (Ritter's method).

76 Assumptions in farm calculations.

77 How is the module of the net system of parallel forces determined?

78 What is the center of parallel forces?

79 What is the center of gravity of the body?

80 Coordinates of the center of the system of parallel forces.

81 Coordinates of the center of gravity of the body.

82 Methods of finding the center of gravity (symmetry).

83 Center of gravity of the volume.

84 Center of gravity of the area.

85 Center of gravity of the line.

86 Integral formulas for determining the coordinates of the center of gravity.

87 Experimental methods of finding the center of gravity (weighing).

88 Center of gravity of the area of the triangle.

89 Center of gravity of the arc of a circle.

90 Center of gravity of the circle sector area.

91 Center of gravity of the volume of a pyramid or cone.

92 Center of gravity of the area of the parallelogram.

93 Methods of finding the center of gravity (supplement).

94 Methods of finding the center of gravity (suspension).

95 Methods of finding the center of gravity (breakdown).

2 KINEMATICS

2.1 Ways to set movement

Kinematics is a branch of mechanics that studies the movement of material bodies from a geometric point of view, without taking into account the mass and the forces acting on them.

Methods for specifying the movement of a point:

1) natural,
2) coordinate,
3) vector.

The trajectory of a point is a continuous curve that the point describes during its movement.

Natural technique the trajectory of the point is indicated, the law of its movement along this trajectory, the beginning and direction of the arc coordinate: s=f(t) - the law of movement of the point. For rectilinear motion: x=f(t).

Coordinate technique the position of a point in space is determined by three coordinates, changes in which determine the law of motion of the point: x=f1(t), y=f2(t), z=f3(t).

If the motion is in a plane, then there are two equations of motion. The equations of motion describe the trajectory equation in parametric form. Eliminating the parameter t from the equations, we obtain the trajectory equation in the usual form: f (x, y) = 0 (for a plane).

Vector technique the position of a point is determined by its radius vector drawn from some center. A curve that is drawn by the end of a vector, called. hodograph of this vector. Those. the trajectory is the hodograph of the radius vector. Relationship between coordinate and vector methods:

$$\vec{r} = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k},$$

($\vec{i}, \vec{j}, \vec{k}$ - orts - unit vectors co-directed with any axis)

modulus

$$r = \sqrt{x^2 + y^2 + z^2},$$

direction cosines:

$$\cos(x,\vec{r}) = \frac{x}{r}, \text{ etc.}$$

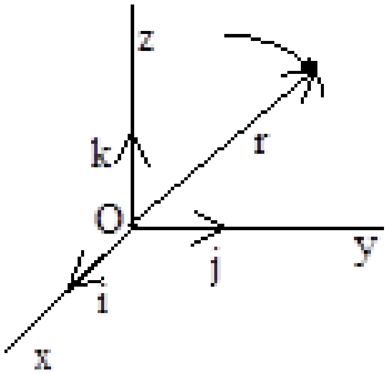


Figure 8

Transition from the coordinate method to the natural one:

$$s = \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt\,.$$

2.2 Point speed

Speed vector- the first derivative of the radius vector with respect to time (the dot denotes the derivative with respect to time);:

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

$$\vec{v} = \frac{dx}{dt}\cdot\vec{i} + \frac{dy}{dt}\cdot\vec{j} + \frac{dz}{dt}\cdot\vec{k}\,.$$

Velocity projections:

$$v_x = \frac{dx}{dt} = \dot{x},\ v_y = \frac{dy}{dt} = \dot{y},\ v_z = \frac{dz}{dt} = \dot{z}\,.$$

Speed module:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}\ ,$$

direction cosines:

$$\cos(x, \vec{v}) = \frac{v_x}{v}, \text{ etc.}$$

If the modulus of speed does not change over time, then the movement is called uniform. With natural technique:

$$v = \frac{ds}{dt} = \dot{s}$$

- velocity modulus, velocity vector:

$$\vec{v} = \frac{ds}{dt} \cdot \vec{\tau} \ ,$$

$\vec{\tau}$- tangent unit vector, i.e. the velocity is always directed tangentially to the trajectory. If v>0, then the movement occurs in the direction of the positive reading of the arc coordinate and vice versa. Movement in the polar coordinate system: r=r(t) – polar radius, φ=φ(t) – angle. Velocity projections on radial direction $v_r = \dot{r}$, transverse direction $v_p = r \cdot \dot{\varphi}$, velocity modulus

$$v = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}\ ; \ x=r\cos\varphi,\ y=r\sin\varphi.$$

2.3 Acceleration point

Point acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \dot{\vec{v}} = \ddot{\vec{r}}, \text{ [m/sec2].}$$

Acceleration projections:

$$a_x = \frac{dv_x}{dt} = \dot{v}_x = \ddot{x}, \text{ etc.}$$

Acceleration modulus:

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}, \qquad \cos(x, \vec{a}) = \frac{a_x}{a}, \text{ etc.}$$

When specifying motion in polar coordinates: acceleration projections on the radial direction $a_r = \ddot{r} - r \cdot \dot{\varphi}^2$, transverse direction $a_p = r \cdot \ddot{\varphi} + 2\dot{r}\dot{\varphi}$, acceleration modulus $a = \sqrt{a_r^2 + a_p^2}$. With natural technique of motion tasks full acceleration are decomposed into normal and tangential (tangential) accelerations:

$$\vec{a} = \vec{a}_n + \vec{a}_\tau.$$

Modulus of normal acceleration:

$$a_n = \frac{v^2}{\rho},$$

ρ is the radius of curvature of the trajectory, normal acceleration is directed along the normal to the trajectory (⊥ to the tangent) always to the center of curvature, i.e. towards the concavity. Normal acceleration characterizes the change in speed in direction. The module of tangential acceleration

$$a_\tau = \frac{dv}{dt} = \frac{d^2s}{dt^2},$$

directed tangentially to the trajectory, either in the direction of speed, or in the opposite direction. Tangential acceleration characterizes the change in speed in magnitude. With accelerated movement, the direction of the tangentially acceleration and the speeds are the same, while slowing down - the opposite.

$$\vec{a}_n \perp \vec{a}_\tau, \quad \Rightarrow \quad a = \sqrt{a_n^2 + a_\tau^2}\,..$$

The acceleration vector lies in the contiguous plane that its projection on the binormal is 0 (the main normal lies in the contiguous plane, i.e. in the plane of the plane curve, the binormal is ⊥ to the main normal and the tangent).

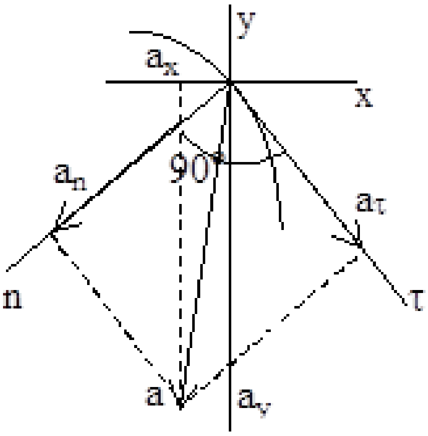


Figure 9

Particular cases of point movement:

1) Rectilinear: radius of curvature $\rho = \infty$ (infinitely large) $\Rightarrow a_n=0$, $a=a_\tau$..

2) Uniform curvilinear motion: v=const $\Rightarrow a_\tau=0$, $a=a_n$. Acceleration appears only due to a change in the direction of speed. Law of motion: $s=s_0+v\cdot t$, at $s_0=0$ $v=s/t$.

3) Uniform rectilinear motion: $a=a_\tau=a_n=0$. The only movement where a=0.

4) Equally variable curvilinear motion:

$$a_\tau=\text{const}, \quad v=v_0+a_\tau\cdot t, \quad s = s_0 + v_0 t + \frac{a_\tau t^2}{2}.$$

With equilateral the signs of a_τ and v are the same, with equally slow motion they are different.

2.4 Translational movement of the body

The simplest movements of a rigid body: translational and rotation around a fixed axis. The translational motion of a body is such a motion of a rigid body, in which any straight line drawn in this body moves, remaining parallel to itself. When acting movement all points of the body describe the same trajectories and at each moment of time have the same magnitude and direction of speed and acceleration.

2.5 Rotational movement of the body

The rotational motion of a body is such a motion of a rigid body, in which all points belonging to a certain straight line, invariably connected with the body, remain motionless. This line is called the axis of rotation of the body. With this movement, all points of the body move in planes perpendicular to the axis of rotation, and describe circles whose centers lie on the axis of rotation. The equation (law) of rotational motion: $\varphi=f(t)$ – body rotation angle in radians. (1 rad= $180^o/\pi=57{,}3^o$).

Angular velocity:

$$\omega = \frac{d\varphi}{dt} = \dot{\varphi}, \text{ [rad/s]}$$

– determines the rate of change of the angle of rotation.

The angular velocity vector of a body rotating around a fixed axis is directed along the axis of rotation so that if you look towards it, the rotation will be counterclockwise. arrow. "n" is the number of revolutions per minute [rpm],

$$1\text{rev}=2\pi \text{ rad}, \ \omega = \frac{\pi n}{30}.$$

Angular acceleration of the body:

$$\varepsilon = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2} = \dot{\omega} = \ddot{\varphi}, \text{ [rad/s}^2\text{]}.$$

The angular acceleration vector is also directed along the rotation axis. With accelerated motion, it coincides in direction with the angular velocity and oppositely with slow rotation.

Particular cases of body rotation:

1) Uniform rotation:

$$\omega=\text{const}, \ \varphi=\omega t, \ \omega=\varphi/t,$$

2) Equal-variable rotation:

$$\omega=\omega_0+\varepsilon t; \quad \varphi=\omega_0 t+\frac{\varepsilon t^2}{2},$$

here the initial angle $\varphi_0=0$.

Velocities and accelerations of points of a rotating body.

$$\vec{v}=\vec{\omega}\times\vec{r}$$

- the speed of any point of a rigid body rotating around a fixed axis is equal to the vector product of the angular velocity vector of the body and the radius vector of this point. The module of the vector product:

$$v=\omega\cdot r\cdot\sin(\alpha)=\omega\cdot(CM),$$

(CM) is the distance from point M to the axis of rotation (fig.10). The velocity vector is directed along the tangent to the circle, along which the point M moves, in the direction of rotation.

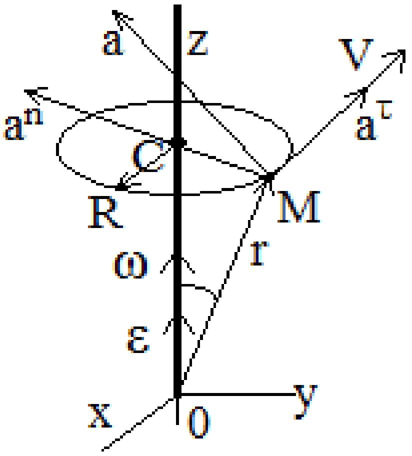


Figure 10

Euler formulas:

$$\vec{v}=\vec{\omega}\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \omega_x & \omega_y & \omega_x\\ x & y & z\end{vmatrix}=\vec{i}(\omega_y z-\omega_z y)+\vec{j}(\omega_z x-\omega_x z)+\vec{k}(\omega_x y-\omega_y x),$$

$\omega_x, \omega_y, \omega_z$ are projections of the angular velocity vector. Projection of rotational (circumferential) speed:

$$v_x=\omega_y z-\omega_z y; \qquad v_y=\omega_z x-\omega_x z; \qquad v_z=\omega_x y-\omega_y x.$$

If the axis of rotation coincides with the z-axis, then

$$v_x=-\omega y; \qquad v_y=\omega x.$$

Acceleration:

$$\vec{a}=\vec{\varepsilon}\times\vec{r}+\vec{\omega}\times\vec{v}=\vec{\varepsilon}\times\vec{r}+\vec{\omega}\times(\vec{\omega}\times\vec{r}).$$

Rotational acceleration

$$\vec{a}^{\tau}=\vec{\varepsilon}\times\vec{r},$$

the module of rotational acceleration

$$a^{\tau}=\varepsilon\cdot r\cdot\sin\alpha,$$

directed tangentially to the trajectory of the point, i.e. parallel to speed. Centripetal (oscillating) acceleration,

$$\vec{a}^{n}=\vec{\omega}\times\vec{v}=\vec{\omega}\times(\vec{\omega}\times\vec{r}), \qquad a^n=\omega^2\cdot R$$

is directed along the radius to the axis (center) of rotation.

Full acceleration module:

$$a=\sqrt{(a^n)^2+(a^{\tau})^2}=R\sqrt{\varepsilon^2+\omega^4}.$$

Angle between the vectors of full and centripetal accelerations:

$$tg\beta=\frac{a^{\tau}}{a^{n}}=\frac{\varepsilon}{\omega^2}.$$

2.6 Planar motion of a rigid body

Plane (plane-parallel) is a movement in which all its points move parallel to some fixed plane. Plane motion equations:

$$x_A = f_1(t), \quad y_A = f_2(t), \quad \varphi = f_3(t),$$

point A is called a pole. The plane motion of a solid body is composed of translational motion, in which all points of the body move in the same way as the pole (A), and of rotational motion around this pole. The translational movement depends on the choice of the pole, and the magnitude and direction of the angle of rotation are independent.

Velocity of points of the body during plane motion:

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{AB}; \qquad \vec{v}_B = \vec{v}_A + \vec{v}_{BA}, \qquad v_{BA} = \omega \cdot BA,$$

i.e. the speed of any point B of a plane figure is equal to the geometric sum of the speed of pole A and the speed of point B during the rotation of a plane figure around pole A.

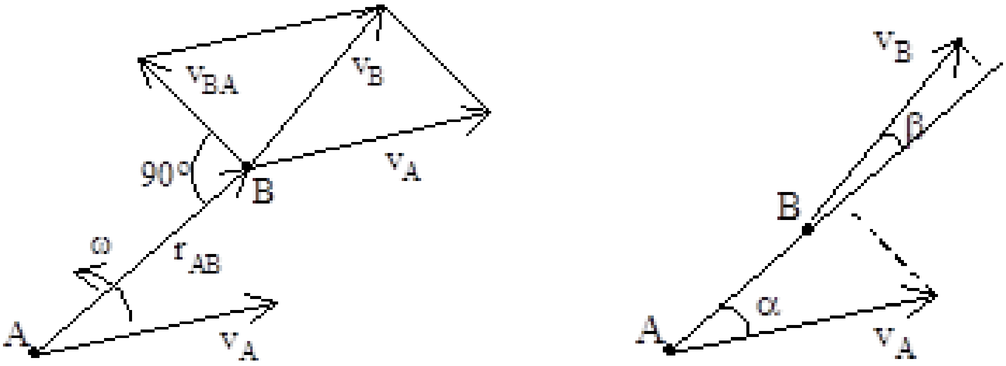


Figure 11

Theorem: with a plane movement, the projections of the velocities of two points of the body on an axis passing through these points are equal to each other: $v_A\cos\alpha = v_B\cos\beta$.

The instantaneous center of velocities is a point of a flat figure, the speed of which is currently equal to zero - P. If the body moves non-translationally, i.e. $\omega \neq 0$, then instantaneous center of velocities (ICV) always exists. With the translational movement of the ICV is located in ∞.

$$\vec{v}_B = \vec{\omega} \times \vec{PB}$$

- the speed of any point of a flat figure has a module equal to the product of the angular velocity of the figure and the length of the segment connecting the point with the ICV, and is directed ⊥ to this segment in the direction of rotation of the figure.

$$\frac{v_A}{PA} = \frac{v_B}{PB},$$

the velocities of the points of the body are proportional to their distances to the ICV

$$\omega = \frac{v_B}{PB},$$

the angular velocity of the body is equal to the ratio of the velocity of some point to its distance to the ICV.

Determining the position of the ICV (fig.12):

1) ICV - point of intersection of perpendiculars restored to the velocities of points (directed at point B and point K);

2) if the velocities of points A and B are parallel to each other and perpendicular to AB, then to determine the ICV the magnitudes and directions of the velocities must be known (see v_A и v_B);

3) if they are equal to each other, then the ICV is in ∞, and the angular velocity $\omega = v_A/\infty = 0$;

4) if it is known that the velocities of two points A and B are equal, parallel and not perpendicular to AB, then the ICV in ∞, and the angular velocity $\omega = v_A/\infty = 0$, if this takes place only at a certain point in time, then we have an instantaneous translational motion;

5) if a flat figure rolls without slipping on a fixed surface, then the ICV flat figure will be at the point of contact.

Chall's theorem: a plane figure can be moved from one position to any other position on the plane by one rotation of this figure around some fixed center. This center on a fixed plane coincides with the ICV and is called the instantaneous center of rotation (axis of rotation). When a flat figure moves, the ICV constantly changes its position. The locus of the ICV marked on the fixed plane is called the fixed centroid. The geometric place of the ICV, marked on the plane of the figure, is called the moving centroid (the wheel rolls in a straight line: the fixed centroid is a straight line, the moving centroid is a circle).

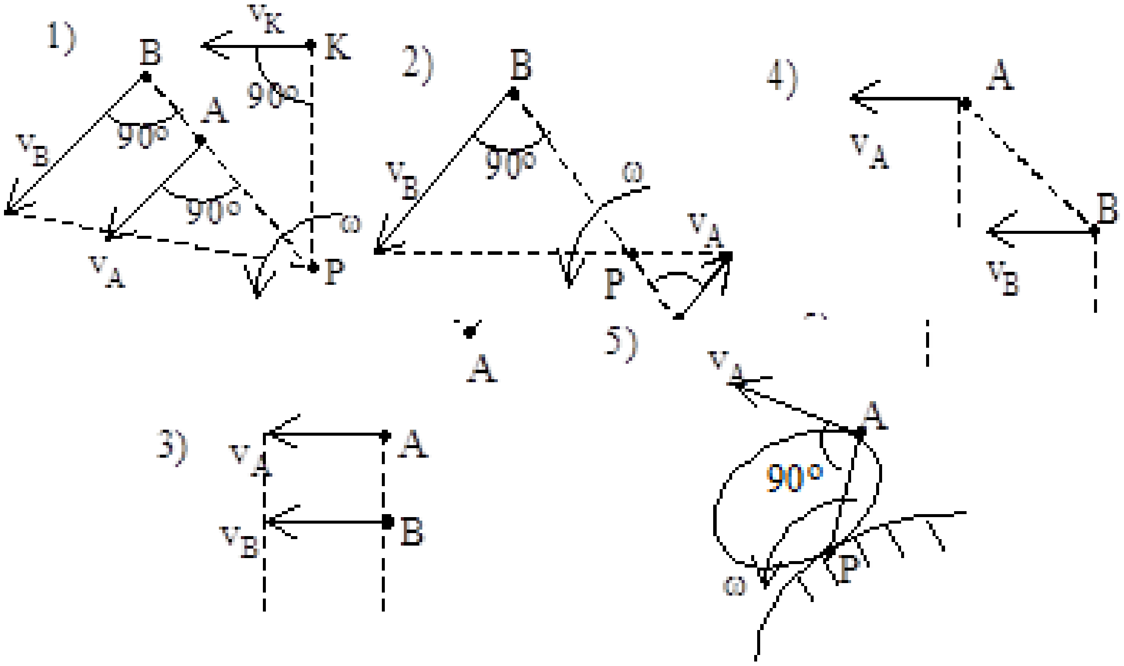


Figure 12

When a flat figure moves, the movable centroid rolls without slipping along the stationary centroid (Poinsot's theorem).

Point accelerations:

$$\vec a_B = \frac{d\vec v_B}{dt} = \frac{d\vec v_A}{dt} + \frac{d\vec \omega}{dt} \times \vec r_{AB} + \vec \omega \times \frac{d\vec r_{AB}}{dt}, \quad \vec a_B = \vec a_A + \vec a_{BA} = \vec a_A + \vec a^{ц}_{BA} + \vec a^{вр}_{BA}$$

– the acceleration of any point (B) of the figure geometrically consists of the acceleration of the pole (A) and the centripetal and rotational accelerations in the rotational motion of the body relative to the pole.

$$a^n_{BA} = \omega^2 \cdot BA, \; a^\tau_{BA} = \varepsilon \cdot BA, \; tg\alpha = \frac{a^\tau_{BA}}{a^n_{BA}} = \frac{\varepsilon}{\omega^2}, \; a_{BA} = AB\sqrt{\varepsilon^2 + \omega^4}\,.$$

The instantaneous center of accelerations is a point (Q) of a flat figure, the acceleration of which at a given moment of time is equal to zero. To build it from point A, we lay off the segment at an angle to the acceleration a_A

$$\alpha = \operatorname{arctg}\frac{\varepsilon}{\omega^2}$$

$$AQ = \frac{a_A}{\sqrt{\varepsilon^2 + \omega^4}},$$

while the angle is postponed from the acceleration to the side, the direction of the angular acceleration ε. Modules of accelerations of points of a flat figure are proportional to the distances from these points to the instantaneous center of accelerations, and the acceleration vectors are composed of segments connecting these points and the instantaneous center of accelerations the same angle. The instantaneous center of velocities P and the instantaneous center of accelerations Q are different points on a plane figure.

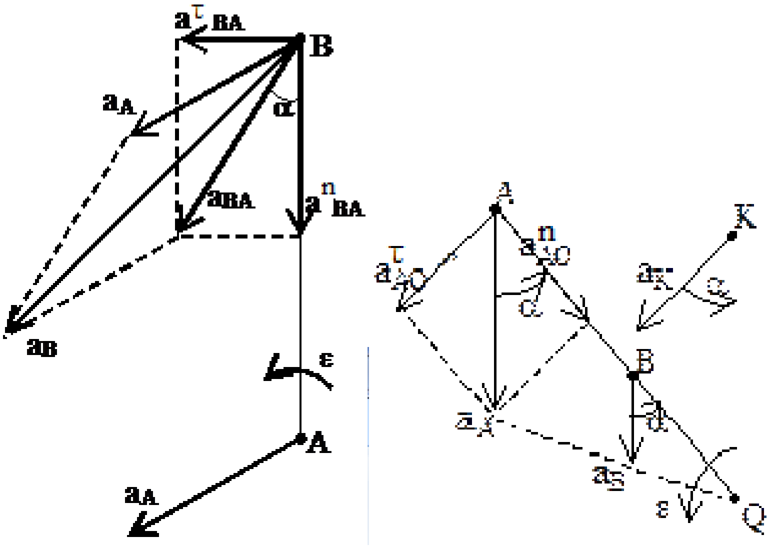


Figure 13

2.7 Spherical motion of a rigid body

Spherical motion is the motion of a rigid body, one of the points of which remains motionless during the entire motion (eg the motion of a top). The points of the body move along spherical surfaces. The position of the body is determined using three angles. To do this, two coordinate systems are set: fixed Oxyz and moving Oξης, connected with the rigid body. Line OJ – line of knots, angles are set: Ψ – precession angle, θ – nutation angle, φ – angle of proper rotation – Euler angles. Thus, the equations of spherical motion:

$$\Psi=f_1(t);\ \theta=f_2(t);\ \varphi=f_3(t).$$

The angles are measured from the axes against the stroke of the clock.

Euler-D'Alembert theorem: any displacement of a body having a fixed point can be replaced by one rotation around some instantaneous axis of rotation passing through this point. The velocities of all points of the body lying on the instantaneous axis of rotation at a given time are equal to zero. The vector of angular velocity (instantaneous angular velocity) is plotted from a fixed point along the instantaneous axis in such a direction that, looking towards this vector, one can see the rotation occurring counterclockwise. The angular velocity vector changes with time not only in numerical value, but also in direction. The end of the vector describes the hodograph of the velocity of the vector $\vec{\omega}$. Angular acceleration:

$$\vec{\varepsilon}=\frac{d\vec{\omega}}{dt}$$

- the speed of the end of the vector $\vec{\omega}$, coincides in direction with the tangent to the hodograph of the angular velocity vector. In the case of spherical motion, in contrast to the case of rotation around a fixed axis, the vector $\vec{\varepsilon}$ does not coincide with the direction $\vec{\omega}$. Velocities of points during spherical motion:

$$\vec{v}=\vec{\omega}\times\vec{r}$$

- vector composition, $\vec{r}$ - radius-vector of a point, drawn from a fixed point, modulus

$$v=\omega r\cdot\sin\alpha=\omega\cdot h,$$

h - distance from the point to the instantaneous axis of rotation.

Euler's formulas:

$$\vec{v}=\vec{\omega}\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \omega_x & \omega_y & \omega_x\\ x & y & z\end{vmatrix}=\vec{i}(\omega_y z-\omega_z y)+\vec{j}(\omega_z x-\omega_x z)+\vec{k}(\omega_x y-\omega_y x).$$

Accelerations:

$$\vec{a}=\vec{\varepsilon}\times\vec{r}+\vec{\omega}\times\vec{v},$$

rotational acceleration

$$\vec{a}^{\tau}=\vec{\varepsilon}\times\vec{r}$$

module of rotational acceleration

$$a^{\tau}=\varepsilon\cdot r\cdot\sin\beta=\varepsilon\cdot h_1,$$

h_1 is the distance from the point to the vector $\vec{\varepsilon}$, directed ⊥ -but the plane passing through the point M and the vector $\vec{\varepsilon}$. Vibrant acceleration

$$\vec{a}^{c}=\vec{\omega}\times\vec{v},\quad a^{c}=\omega^2\cdot h,$$

is directed to the axis of rotation.

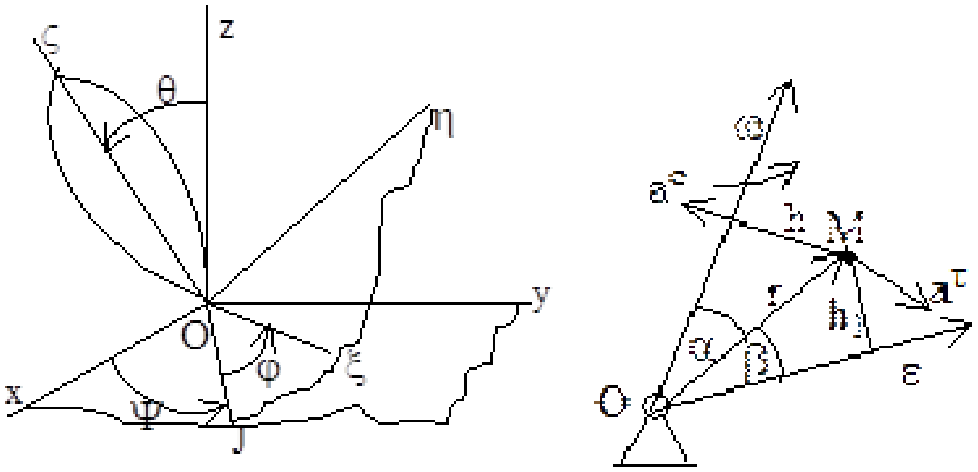


Figure 14

2.8 Motion of a free rigid body

Motion of a free rigid body (general case of motion). A free rigid body has six degrees of freedom. When considering the motion of a free rigid body, in addition to the fixed coordinate system Oxyz, a moving coordinate system $Ax_1y_1z_1$ is introduced, which is associated with the body at point A. Then the motion of a free rigid body is a complex motion that can be considered as consisting from translational movement along with the pole (A) and spherical movement around the pole. Equations of motion of a free rigid body:

$$x_A=f_1(t);\ y_A=f_2(t);\ z_A=f_3(t);$$

Euler angles:

$$\Psi=f_4(t);\ \theta=f_5(t);\ \varphi=f_6(t)$$

The first three equations determine the translational part of the movement and depend on the choice of the pole, the remaining three are determined by the spherical movement around the pole and do not depend on the choice of the pole. The speed of any point of a free rigid body equal the geometric sum of the speed of the pole and the speed of this point in its spherical motion around the pole.

$$\vec{v} = \vec{v}_A + \vec{\omega} \times \vec{r}$$

Acceleration of a point of a free rigid body equal the geometric sum of the acceleration of the pole, the sharp acceleration of the point and its rotational acceleration, defined relative to the instantaneous axis and the axis of angular acceleration passing through the pole.

$$\vec{a} = \vec{a}_A + +\vec{a}^{oc} + \vec{a}^{вр} = \vec{a}_A + +\vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\varepsilon} \times \vec{r} \ ,$$

the last two terms give the acceleration of the point in its motion around the pole.

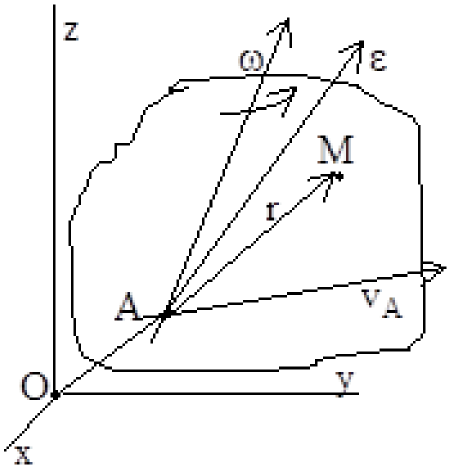


Figure 15

2.9 Complicated point movement

Complex movement of a point (body). A complex movement of a point (body) is a movement in which a point (body) simultaneously participates in several movements (for example, a passenger moving along a moving car). In this case, a moving coordinate system (Oxyz) is introduced, which performs a given movement relative to the fixed (main) coordinate system ($O_1x_1y_1z_1$).

The absolute movement of the point is called. movement with respect to a fixed coordinate system. Relative motion is motion relative to a moving coordinate system (movement on the car). Portable movement - the movement of a mobile system coordinates relative to the fixed one (car movement).

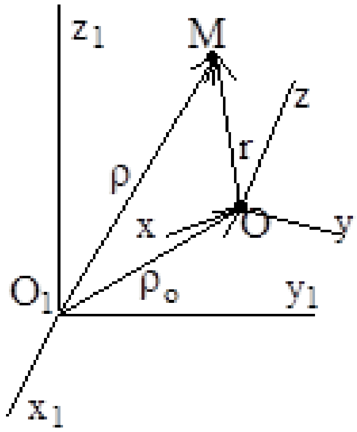


Figure 16

Velocity addition theorem:

$$\vec{\rho} = \vec{\rho}_O + \vec{r}\,,\ \ \vec{v} = \frac{d\vec{\rho}}{dt} = \frac{d\vec{\rho}_O}{dt} + \left(\frac{d\vec{i}}{dt}x + \frac{d\vec{j}}{dt}y + \frac{d\vec{k}}{dt}z\right) + \left(\vec{i}\frac{dx}{dt} + \vec{j}\frac{dy}{dt} + \vec{k}\frac{dz}{dt}\right);$$

$\vec{i}, \vec{j}, \vec{k}$-orts (unit vectors) of the moving coordinate system, the ort rotates around the instantaneous axis, so the speed of its end

$$\frac{d\vec{i}}{dt} = \vec{\omega}_e \times \vec{i}\,, \text{ etc., : } \quad \vec{v} = \frac{d\vec{\rho}_O}{dt} + \vec{\omega}_e \times (\vec{i}x + \vec{j}y + \vec{k}z) + \left(\vec{i}\frac{dx}{dt} + \vec{j}\frac{dy}{dt} + \vec{k}\frac{dz}{dt}\right),$$

$$\vec{i}x + \vec{j}y + \vec{k}z = \vec{r}\,; \qquad \vec{i}\frac{dx}{dt} + \vec{j}\frac{dy}{dt} + \vec{k}\frac{dz}{dt} = \vec{i}v_{rx} + \vec{j}v_{ry} + \vec{k}v_{rz} = \vec{v}_r$$

- relative speed.

$$\vec{v} = \vec{v}_O + \vec{\omega}_e \times \vec{r} + \vec{v}_r\,;$$

portable speed:

$$\vec{v}_e = \vec{v}_O + \vec{\omega}_e \times \vec{r}\,,$$

therefore, the absolute speed of a point equal the geometric sum of its portable (v_e) and relative (v_r) speeds

$$\vec{v} = \vec{v}_e + \vec{v}_r\,, \qquad v = \sqrt{v_e^2 + v_r^2 + 2v_e v_r \cos(\vec{v}_e, \vec{v}_r)}\,.$$

Acceleration addition theorem (Coriolis theorem):

$$\vec a = \frac{d\vec v}{dt} = \frac{d^2\rho_O}{dt^2} + (\frac{d^2\vec i}{dt^2}x + \frac{d^2\vec j}{dt^2}y + \frac{d^2\vec k}{dt^2}z) +$$

$$+(\vec i\frac{d^2x}{dt^2} + \vec j\frac{d^2y}{dt^2} + \vec k\frac{d^2z}{dt^2}) + 2(\frac{d\vec i}{dt}\frac{dx}{dt} + \frac{d\vec j}{dt}\frac{dy}{dt} + \frac{d\vec k}{dt}\frac{dz}{dt})$$

$$\frac{d^2\vec i}{dt^2} = \frac{d}{dt}(\frac{d\vec i}{dt}) = \frac{d}{dt}(\vec\omega_e \times \vec i) = \frac{d\vec\omega_e}{dt} \times \vec i + \omega_e \times \frac{d\vec i}{dt} = \vec\varepsilon_e \times \vec i + \omega_e \times (\omega_e \times \vec i)\text{, etc.}$$

The terms of the expression that determines the acceleration:

1) $\frac{d^2\rho_O}{dt^2} = \vec a_O$ - the acceleration of the pole O;

2)

$$\frac{d^2\vec i}{dt^2}x + \frac{d^2\vec j}{dt^2}y + \frac{d^2\vec k}{dt^2}z + [\vec\varepsilon_e \times \vec i + \vec\omega_e \times (\vec\omega_e \times \vec i)] \cdot x + [\vec\varepsilon_e \times \vec j + \vec\omega_e \times (\vec\omega_e \times \vec j)] \cdot y +$$

$$+ [\vec\varepsilon_e \times \vec k + \vec\omega_e \times (\vec\omega_e \times \vec k)] \cdot z = \vec\varepsilon_e \times \vec r + \vec\omega_e \times (\vec\omega_e \times \vec r);$$

3) $\vec i\frac{d^2x}{dt^2} + \vec j\frac{d^2y}{dt^2} + \vec k\frac{d^2z}{dt^2} = \vec i \cdot a_{rx} + \vec j \cdot a_{ry} + \vec k \cdot a_{rz} = \vec a_r$

is the relative acceleration of the point;

4)

$$\frac{d\vec i}{dt}\frac{dx}{dt} + \frac{d\vec j}{dt}\frac{dy}{dt} + \frac{d\vec k}{dt}\frac{dz}{dt} = \vec\omega_e \times (\vec i\frac{dx}{dt} + \vec j\frac{dy}{dt} + \vec k\frac{dz}{dt}) = \vec\omega_e \times (\vec i v_{rx} + \vec j v_{ry} + \vec k v_{rz}) = \vec\omega_e \times \vec v_r,$$

we get:

$$\vec a = \vec a_O + \vec\varepsilon_e \times \vec r + \vec\omega_e \times (\vec\omega_e \times \vec r) + \vec a_r + 2(\vec\omega_e \times \vec v_r).$$

The first three terms represent the acceleration of a point in a portable motion: $\vec a_O$ - the acceleration of the pole O;

$$\vec a_e^{вр} = \vec\varepsilon_e \times \vec r$$

- rotational acceleration,

$$\vec{a}_e^{oc} = \vec{\omega}_e \times (\vec{\omega}_e \times \vec{r})$$

- sharp acceleration, i.e.

$$\vec{a}_e = \vec{a}_O + \vec{a}_e^{вр} + \vec{a}_e^{oc} \ .$$

The theorem on the addition of accelerations (Coriolis theorem):

$$\vec{a} = \vec{a}_e + \vec{a}_r + \vec{a}_c ,$$

$$\vec{a}_c = 2\vec{\omega}_e \times v_r$$

- Coriolis acceleration (Coriolis acceleration) - in the case of non-translational translational motion, absolute acceleration equal the geometric sum of translational, relative and Coriolis accelerations. Coriolis acceleration characterizes:

1) a change in the module and direction of the portable velocity of a point due to its relative motion;

2) change in the direction of the relative velocity of the point due to rotational translational motion. Coriolis acceleration modulus:

$$a_c = 2 \cdot |\omega_e \cdot v_r| \cdot \sin(\omega_e \hat{} v_r),$$

the direction of the vector is determined by the vector product rule, or by the Zhukovsky rule: the projection of the relative velocity onto a plane perpendicular to the translational angular velocity , must be rotated 90° in the direction of rotation.

Coriolis accelerations equal 0 in three cases:

1) $\omega_e = 0$, i.e. in the case of translational portable motion or at the moment of circulation of the angle. speed to 0;

2) $v_r = 0$;

3) $\sin(\omega_e \hat{} v_r) = 0$, i.e. $\angle(\omega_e \hat{} v_r) = 0$ when the relative velocity v_r is parallel to the translational rotation axis. In the case of movement in one plane, the angle between v_r and the vector $\omega_e = 90^o$, $\sin 90^o = 1$, $a_c = 2 \cdot \omega_e \cdot v_r$.

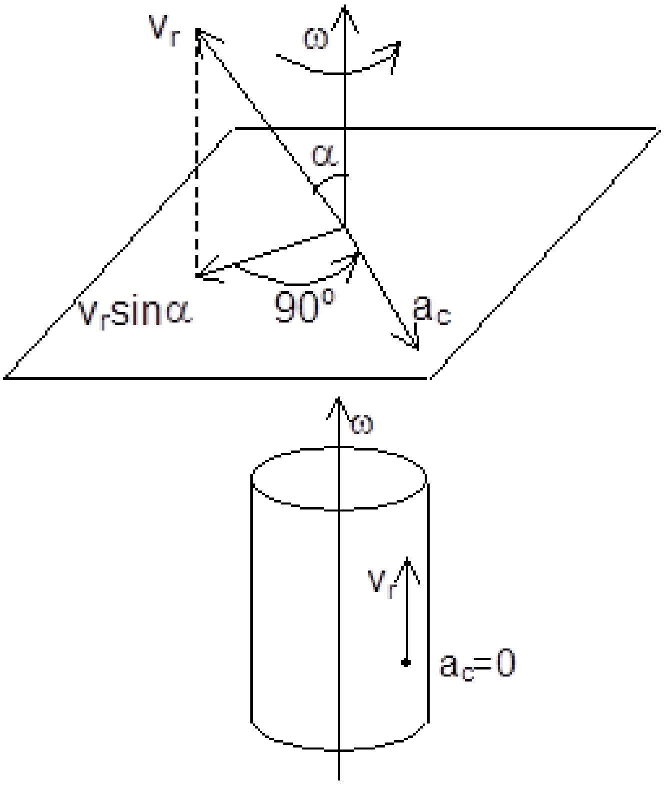


Figure 17

2.10 Complex motion of a rigid body

When adding two translational movements, the resulting movement is also translational and the speed of the resulting movement is equal to the sum of the speeds of the constituent movements.

Addition of rotations of solid bodies around intersecting axes. The axis of rotation, the position of which in space changes with time is called. instantaneous axis of rotation of the body. The angular velocity vector is a sliding vector directed along the instantaneous axis of rotation.

The absolute angular velocity of the body = the geometric sum of the velocities of the constituent rotations - the rule of the parallelogram of angular velocities.

$$\vec{\omega} = \vec{\omega}_e + \vec{\omega}_r .$$

If the body participates simultaneously in instantaneous rotations around several axes intersecting at one point, then

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \ldots + \vec{\omega}_n \ .$$

With the spherical motion of a rigid body, one of the points of which remains motionless during the entire motion, we have the equations of spherical motion:

$$\Psi=f_1(t);\ \theta=f_2(t);\ \varphi=f_3(t).$$

Ψ – precession angle,
θ – nutation angle,
φ – angle of proper rotation – Euler angles.
Angular velocity of precession

$$\omega_1 = \frac{d\Psi}{dt},$$

angular nutation rate

$$\omega_2 = \frac{d\theta}{dt},$$

angular velocity own rotation

$$\omega_3 = \frac{d\varphi}{dt}.\ \ \vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3,$$

$$\omega = \sqrt{\dot{\Psi}^2 + \dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\Psi}\dot{\varphi}\cos\theta}$$

is the modulus of the angular velocity of the body around the instantaneous axis. Via projections onto fixed coordinate axes – kinematic Euler equations:

$$\omega_x = \dot{\theta}\cos\Psi + \dot{\varphi}\sin\theta\sin\Psi;\ \ \omega_y = \dot{\theta}\sin\Psi - \dot{\varphi}\sin\theta\cos\Psi;\ \ \omega_z = \dot{\Psi} + \dot{\varphi}\cos\theta$$

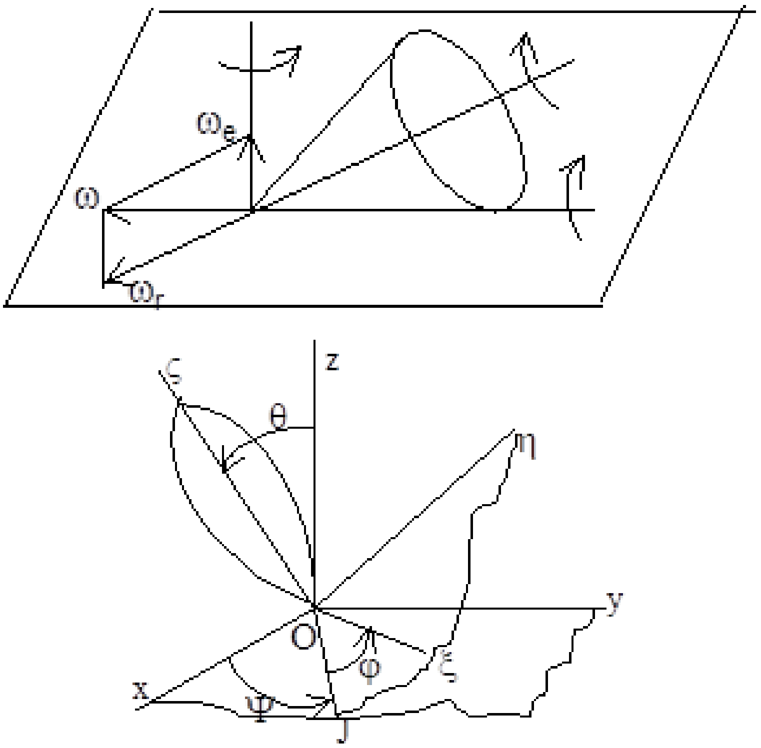


Figure 18

Addition of rotations around 2 parallel axes.

1) Rotations are directed in one direction. $\omega=\omega_2+\omega_1$, C is the instantaneous center of velocities and the instantaneous axis of rotation passes through it,

$$\omega=\frac{v_B}{BC}=\frac{v_A}{AC}==\frac{v_A+v_B}{AB}, \qquad \frac{\omega_1}{BC}=\frac{\omega_2}{AC}=\frac{\omega}{AB}.$$

2) Rotations are directed in different directions.

$$\omega=\frac{v_B}{BC}=\frac{v_A}{AC}==\frac{v_B-v_A}{AB}, \qquad \omega=\omega_2—\omega_1$$

C - instantaneous center of velocities and instantaneous axis of rotation,

$$\frac{\omega_1}{BC}=\frac{\omega_2}{AC}=\frac{\omega}{AB}.$$

The vectors of angular velocities during rotation around parallel axes are added in the same way as the vectors of parallel forces.

3) A pair of rotations - rotations around parallel axes are directed in different directions and the angular velocities are equal in absolute value ($\vec{\omega}_2 = -\vec{\omega}_1$ - a pair of angular velocities). In this case, $v_A=v_B$, the resulting motion of the body is translational (or instantaneous translational) motion at a speed

$$v=\omega_1 \cdot AB$$

- the moment of a pair of angular velocities (translational motion of the bicycle pedal relative to the frame). The instantaneous center of velocities is at infinity.

Addition of translational and rotational movements.

1) The speed of translational movement perpendicular to the axis of rotation - plane-parallel movement - instantaneous rotation around the axis Pp with an angular velocity ω=ω'.

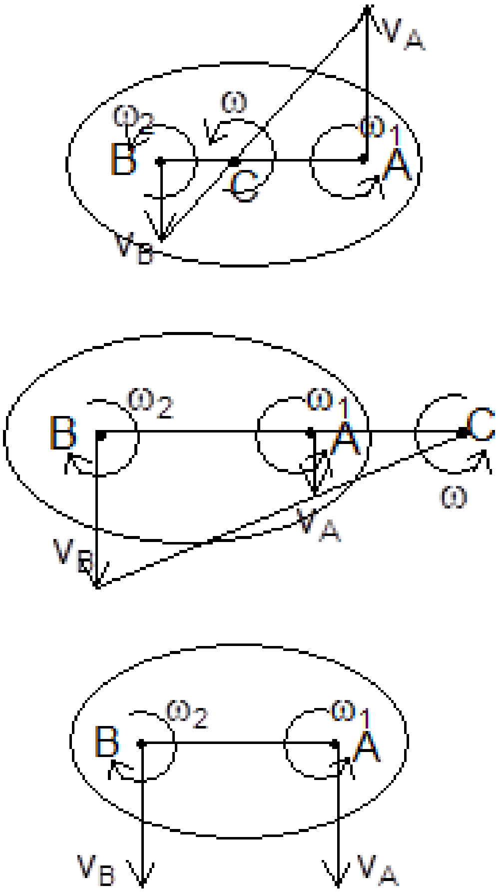


Figure 19

2) Screw movement - the movement of the body is composed of rotational movement around the Aa axis with angular speed ω and translational with speed v parallel Aa. The axis Aa is the axis of the screw. If v and ω are in the same direction, then the screw is right; if they are different, it is left. The distance traveled during one revolution by any point of the body lying on the axis of the screw, called screw pitch - h.

If v and ω are constant, then

$$h = 2\pi \frac{v}{\omega} = \text{const},$$

with a constant pitch, any point M not lying on the axis of the screw describes a helix.

$$v_M = \sqrt{v^2 + \omega^2 r^2}$$

Velocity directed tangentially to the helix.

3) The speed of translational motion forms an arbitrary angle with the axis of rotation, in this case the motion can be considered as a sum of a series of instantaneous screw motions, around continuously changing screw axes - instantaneous screw motion.

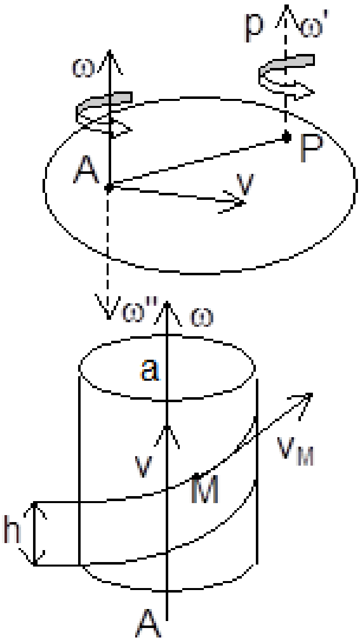


Figure 20

CONTROL QUESTIONS

1 Kinematics (definition). Problem of kinematics.
2 What is the law of motion of a point?
3 The law of motion of a point in vector form.
4 The law of motion of a point in coordinate form.
5 The law of motion of a point in natural form.
6 Interrelationship between different forms of the point movement task.
7 Velocity of a point in the vector method of the task of moving a point.
8 The speed of a point in the coordinate method of the task of moving a point.
9 The speed of a point in the natural method of the task of moving a point.
10 What is the trajectory of a point?
11 Uniform motion of a point. The law of uniform motion of a point.
12 Constant motion of a point. The law of constant motion of a point.
13 What is the speed of a point at a given time?
14 What is the acceleration of a point at a given time?
15 Acceleration of a point in the vector method of the task of moving a point.
16 Acceleration of a point in the coordinate method of the task of moving a point.
17 Acceleration of a point in the natural method of tasking the movement of a point.
18 Normal and tangential acceleration of a point.
19 The radius of curvature and the law of motion of a point along a trajectory.
20 What is the translational movement of the body? The law of transition body movements
21 Theorem on translational motion of a body.
22 Conclusions about translational movement of the body.
23 What is the rotational movement of the body? The law of rotation body movements
24 Angular velocity during rotational motion of the body.
25 Angular acceleration during rotational motion of the body.
26 The relationship between rotation frequency and angular velocity.
27 Uniform rotation of the body (definition and law).
28 Uniform rotation of a body (definition and law).
29 Angular velocity and angular acceleration as a vector.

30 Velocity of body points in rotational motion.
31 Acceleration of body points in rotational motion.
32 Velocity vector of a body point during rotational motion (formula Euler).
33 Vector of tangential acceleration of a point of the body during rotation move
34 Vector of normal acceleration of a point of the body during rotation move
35 What is the gear ratio?
36 What is the planar movement of the body? The law of planar body motion.
37 Decomposition of planar body motion.
38 How the component parts of the plane motion of a figure change when changing poles?
39 Theorem about the projections of the velocities of two points of the body.
40 Theorem on the velocity of a point of a plane figure.
41 What is the instantaneous center of velocity (ICV)?
42 How to find the position of the instantaneous center of velocity?
43 Determining the velocities of points with the help of ICV.
44 Individual cases of the plane move.
45 Speed plan. The purpose of building a speed plan.
46 Determination of the modulus and direction of the velocities of points using speed plan.
47 Determination of the modulus and direction of angular velocities of mechanical links using the speed plan.
48 Theorem on acceleration of points of a plane figure.
49 What is the instantaneous center of acceleration (ICA)?
50 Determination of the position of the ICA.
51 Determining accelerations of points of a flat figure using ICA.
52 Acceleration plan. Purpose of building an acceleration plan.
53 Determination of the modulus and intensity of the accelerations of the points of a flat figure.
54 Determination of the modulus and direction of angular accelerations of mechanical links with the help of an acceleration plan.
55 Parameters of complex movement of a point.
56 Theorem on determining the velocities of a point in complex motion.

57 Addition of accelerations during the translational transfer movement of a point.
58 Coriolis theorem.
59 Poisson formulas. (Derived from unit vectors of moving coordinate system).
60 Coriolis acceleration (vector and module).
61 The direction of the Coriolis acceleration vector (according to the vector rule product of two vectors).
62 The direction of the Coriolis acceleration vector (according to Zhukovsky's rule).
63 When is the Coriolis acceleration equal to zero?
64 Spherical motion of a rigid body. Definition. Euler's angles. Law of sphere annual movement of the body.
65 Velocity of body points during spherical motion.
66 Acceleration of body points during spherical motion.
67 Adding translational movements of a rigid body.
68 Addition of rotational movements of a rigid body.

3 DYNAMICS

3.1 Basic laws of mechanics

Dynamics is a branch of mechanics that studies the laws of motion of material bodies under the action of forces.

The basic laws of mechanics (the laws of Galileo-Newton):

the law of inertia (1st law Newton): a material point maintains a state of rest or uniform rectilinear motion until the action of other bodies changes this state;

the basic law of dynamics (2nd law Newton): the acceleration of the mother point is proportional to the force applied to it and has the same direction as it $m\vec{w} = \vec{F}$;

the law of equality of action and reaction (3rd law Newton): every action corresponds to an equal and oppositely directed reaction;

the law of independence of forces: several forces simultaneously acting on the mother point impart to the point such an acceleration that one force equal to their geometric sum would tell it.

In classical mechanics, the mass of a moving body is assumed to be equal to the mass of a body at rest, a measure of the body's inertia and its gravitational properties. Mass is equal to the weight of the body divided by the free fall acceleration.

$$m=G/g, \qquad g\approx 9.81 m/s^2.$$

g depends on the geographical latitude of the place and the height above sea level - not a constant value. Force - 1N (Newton), $1 kg\cdot m / s^2$. The frame of reference in which the 1st and 2nd laws are manifested is called the inertial frame of reference. Differential equations of motion of a material point:

$$m \cdot \vec{a} = \sum \vec{F}_i ,$$

in the projection onto the Cartesian coordinate axes:

$$m \cdot \ddot{x} = \sum F_{ix} ; \; m \cdot \ddot{y} = \sum F_{iy} ; \; m \cdot \ddot{z} = \sum F_{iz} ,$$

on the axis of a natural trihedron:

$$ma_\tau=\sum F_{i\tau}; \quad ma_n=\sum F_{in}; \quad ma_b=\sum F_{ib}$$

($a_b=0$ is the projection of the acceleration onto the binormal), i.e.

$$m\frac{d^2S}{dt^2}=\sum F_{i\tau}; \; m\frac{V^2}{\rho}=\sum F_{in}; \; 0=\sum F_b$$

(ρ is the radius of curvature of the trajectory at the current point). In the case of a plane motion of a point in polar coordinates:

$$m(\ddot{r}-r\dot{\varphi}^2)=F_r, \quad \frac{m}{r}\frac{d}{dt}(r^2\dot{\varphi})=F_\varphi.$$

Two main tasks of dynamics: the first task of dynamics is, knowing the law of motion of a point, to determine the force acting on it; the second task of dynamics (the main one) - knowing the forces acting on the point, determine the law of motion of the point.

$$m\frac{d^2x}{dt^2}=\sum F_{ix}$$

- differential equation of rectilinear motion of a point. Integrating it twice, we find the general solution $x=f(t, C_1, C_2)$.

The integration constants C_1, C_2 are sought from the initial conditions: $t=0$, $x=x_0$, $\dot{x}=V_x=V_0$, $x=f(t, x_0, V_0)$ - a particular solution - the law of point motion.

3.2 Oscillatory motion of a material point

Restoring force (elasticity force)

$$F_x=-cx,$$

the force tends to return the point to the equilibrium position, "c" – spring stiffness coefficient equal elastic force at a deformation equal to one [N/m]. Free vibrations

$$m \cdot \ddot{x} = -cx;$$

denoting $c/m=k^2$, we get

$$\ddot{x} + k^2 x = 0$$

– linear homogeneous second-order differential equation, characteristic equation:

$$z^2 + k^2 = 0,$$

its roots are imaginary, the general solution of the differential equation will be

$$x = C_1 \cos kt + C_2 \sin kt,$$

C_1, C_2 are constants integration.

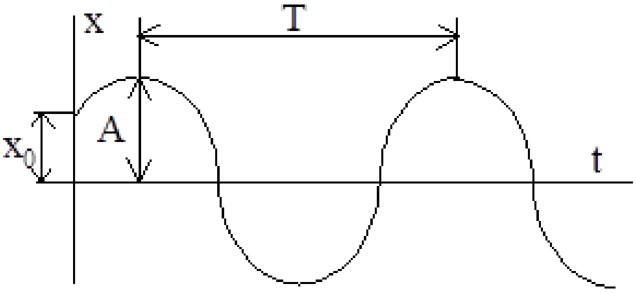


Figure 21

To determine them (C_1, C_2), we find the equation of velocities:

$$\dot{x} = -kC_1 \sin kt + kC_2 \cos kt,$$

we substitute the initial conditions into the equations for x and $\dot{x}$, whence

$$C_1 = x_0, \quad C_2 = \dot{x}_0/k, \quad \text{i.e.} \quad x = x_0 \cos kt + (\dot{x}_0/k)\sin kt.$$

It is possible to designate C_1=Asinβ, C_2=Acosβ,

$$x=A\sin(kt+\beta)$$

– the equation of harmonic oscillations. Amplitude:

$$A=\sqrt{x_0^2+(\dot{x}_0/k^2)}$$

$$tg\beta=kx_0/\dot{x}_0 \ ,$$

β – initial phase of free oscillations;

$$k=\sqrt{c/m}$$

– cyclic frequency (angular, natural) oscillations; period:

$$T=2\pi/k=2\pi\sqrt{m/c}\,,$$

k and T do not depend on the initial conditions – oscillation isochronism; the amplitude and the initial phase depend on the initial conditions. Under the action of a constant force P, the center of oscillations is shifted towards the action of the force P by the value of the static deviation $\delta_{ст}$=P/c. If P is gravity, then

$$T=2\pi\sqrt{\delta_{ст}/g}\,.$$

Damped oscillations under the action of $R_X = -b\dot{x}$ drag force proportional to the speed (viscous friction).

$$m\cdot\ddot{x}=-cx-b\dot{x}\,,$$

denoting b/m=2n, we get:

$$\ddot{x}+2n\dot{x}+k^2x=0 \ ,$$

characteristic equation:

$$z^2 + 2nz + k^2 = 0, \text{ its roots:}$$
$$z_{1,2} = -n \pm \sqrt{n^2 - k^2}\ .$$

1) For n<k imaginary roots than the general solution of the diferencial equation is:

$$x = e^{-nt}(C_1 \cos\sqrt{k^2 - n^2}\,t + C_2 \sin\sqrt{k^2 - n^2}\,t)\,.$$

$$C_1 = A\sin\beta, \quad C_2 = A\cos\beta \quad \Rightarrow \quad x = Ae^{-nt}\sin(kt+\beta).$$

The multiplier e^{-nt} shows that the oscillations are damped. The graph is enclosed between two curves $x = \pm Ae^{-nt}$ symmetrical about the t axis. From the initial conditions:

$$A = \sqrt{x_0^2 + \frac{(\dot{x}_0 + nx_0)^2}{k^2 - n^2}}\,,\ tg\beta = \frac{x_0\sqrt{k^2 - n^2}}{\dot{x} + nx_0}\,;$$

frequency of damped oscillations: $k^* = \sqrt{k^2 - n^2}$; period:

$$T^* = \frac{2\pi}{k^*} = \frac{T}{\sqrt{1 - (n/k)^2}}\,,$$

the period of damped oscillations is greater than the period of free oscillations (with small resistances $T^* \approx T$). Oscillation amplitudes decrease: $\frac{A_{i+1}}{A_i} = e^{-nT^*/2}$ – oscillation decrement; $-nT^*/2$ logarithmic decrement; "n" – attenuation factor.

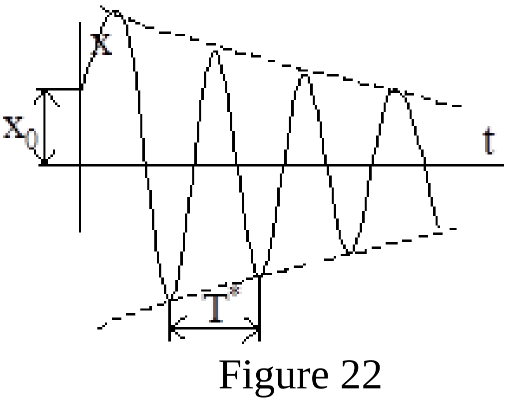


Figure 22

Aperiodic motion of a point at $n \geq k$ or $b \geq 2\sqrt{mc}$. For $n > k$, the roots of the characteristic equation are real, the general solution is:

$$x = e^{-nt}(C_1 e^{\sqrt{n^2-k^2}t} + C_2 e^{\sqrt{n^2-k^2}t}),$$

$$C_1=(B_1+B_2)/2, \quad C_2=(B_1-B_2)/2, \quad x = e^{-nt}(B_1 \mathrm{ch}\sqrt{n^2-k^2}t + B_2 \mathrm{sh}\sqrt{n^2-k^2}t).$$

$$B_1= A\mathrm{sh}\beta, \qquad B_2= A\mathrm{ch}\beta, \qquad x = A \cdot e^{-nt}\mathrm{sh}(\sqrt{n^2-k^2}t+\beta)$$

- this is the equation of non-oscillatory motion (aperiodic), because the hyperbolic sine is not a periodic function. For $n = k$, the roots of the characteristic equation are real, equal and negative: $z_1=z_2= - n$, the general solution:

$$x = e^{-nt}(C_1 t + C_2), \quad \text{or} \quad x = e^{-nt}[x_0 + (\dot{x}_0 + nx_0)t],$$

the motion is also aperiodic.

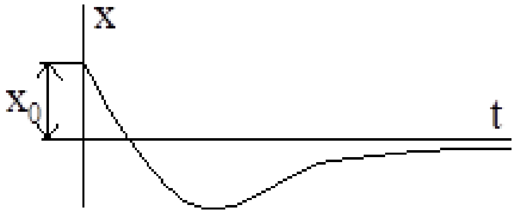


Figure 23

Forced vibrations.

Forced vibrations, in addition to the restoring force, a variable disturbing force acts, usually according to the harmonic law:

$$Q = H\sin(pt + \delta),$$

p is the frequency of the disturbing force, δ is the initial phase.

$$m \cdot \ddot{x} = -cx + H\sin(pt+\delta), \quad h=H/m, \quad \ddot{x} + k^2 x = h\sin(pt+\delta)$$

– differential equation of forced oscillations (non-homogeneous linear differential equation). Its general solution equal the sum of the general solution of the homogeneous equation

$$\ddot{x} + k^2 x = 0$$

and the particular solution of this equation:

$$x = x^{*} + x^{**}.$$

$$x^{*} = C_1 \cos kt + C_2 \sin kt,$$

$$x^{**} = A \sin(pt + \delta)$$

– a particular solution is sought in a form similar to the right side of the equation. Substituting the solution into the equation, we find

$$A = \frac{h}{k^2 - p^2}, \qquad x = C_1 \cos kt + C_2 \sin kt + \frac{h}{k^2 - p^2} \sin(pt + \delta).$$

The value of the static deviation:

$$A_{\text{ст}} = H/c, \; \mu = \frac{A}{A_{\text{ст}}} = \frac{1}{\left|1 - p^2 / k^2\right|}$$

- dynamic coefficient, how many times the oscillation amplitude exceeds the static deviation.

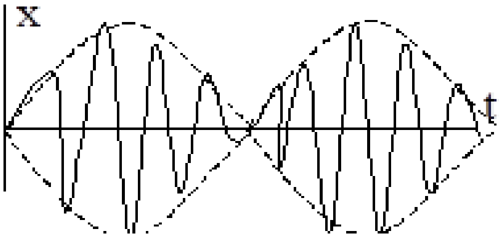


Figure 24

At p=k $\mu = \infty$ – the phenomenon of resonance (the frequency of the disturbing force is equal to the frequency of natural oscillations, while the amplitude increases indefinitely). At p/k≈1, a phenomenon occurs, which is called beats:

$$x = \frac{2h}{k^2 - p^2}\sin(\frac{p-k}{2}t)\cos(pt+\delta).$$

Denoting

$$A = \frac{2h}{k^2 - p^2}\sin(\frac{p-k}{2}t),$$

we get

$$x=A(t)\cos(pt+\delta)$$

– there is an imposition of additional oscillations caused by the perturbing force on the actually forced oscillations – frequency oscillations p, the amplitude of which is a periodic function.

The phenomenon of resonance occurs when the frequencies of forced and free oscillations of the point p=k coincide. Differential equation:

$$\ddot{x} + k^2x = h\sin(kt+\delta).$$

Private solution:

$$x^{**}= Bt\cos(kt+\delta),\ B=-h/(2k),$$

i.e. general solution of the differential equation:

$$x = C_1\cos kt + C_2\sin kt - h/(2k)t\cos(kt+\delta).$$

The equation shows that the amplitude of forced oscillations at resonance increases in proportion to time. Period T=2π/k, the phase of forced oscillations lags behind the phase of the disturbing force by π/2.

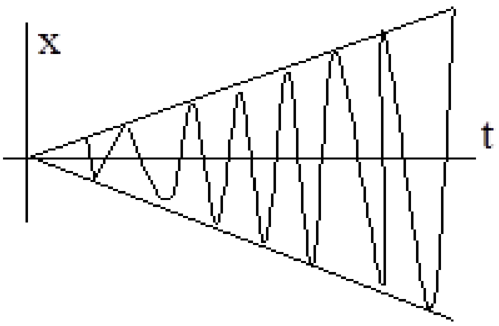


Figure 25

Forced oscillations in the presence of viscous friction:

$$m\cdot\ddot{x} = -cx - b\dot{x} + H\sin(pt+\delta), \qquad \ddot{x} + 2n\dot{x} + k^2x = h\sin(pt+\delta),$$

general solution depending on the value of k and n:
1) for n<k

$$x = Ae^{-nt}\sin(\sqrt{k^2 - n^2}\,t + \beta) + \frac{h}{\sqrt{(k^2 - p^2)^2 + 4n^2p^2}}\sin(pt + \delta - \varepsilon);$$

2) for n>k

$$x = Ae^{-nt}\mathrm{sh}(\sqrt{n^2 - k^2}\,t + \beta) + \frac{h}{\sqrt{(k^2 - p^2)^2 + 4n^2p^2}}\sin(pt + \delta - \varepsilon);$$

3) for n=k

$$x = e^{-nt}(C_1t + C_2) + \frac{h}{\sqrt{(k^2 - p^2)^2 + 4n^2p^2}}\sin(pt + \delta - \varepsilon).$$

3.3 General theorems of point dynamics

3.3.1 Theorem on the change in the momentum of a material point

$$\vec{Q} = m\vec{v}$$

- the amount of motion of a material point, $\vec{F}dt$ - an elementary impulse of force.

$$d(m\vec{v}) = \vec{F}dt$$

- an elementary change in the momentum of a material point is equal to the elementary impulse of the force applied to this point (theorem in differential form) or

$$\frac{d(m\vec{v})}{dt} = \vec{F}$$

- the time derivative of the momentum of the material point is equal to the resultant of the forces applied to this point. We integrate:

$$m\vec{v}_1 - m\vec{v}_0 = \int_0^t \vec{F}dt$$

- the change in the momentum of a material point over a finite period of time is equal to the elementary impulse of the force applied to this point over the same period of time.

$$\vec{S} = \int_0^t \vec{F}dt$$

is the force impulse over the time interval [0,t]. In projections on the coordinate axes:

$$m\dot{x} - m\dot{x}_0 = \int_0^t F_x dt, \text{ etc.}$$

3.3.2 The theorem on the change in the angular momentum of a point

$$\vec{K}_O = \vec{r} \times m\vec{v}$$

- angular momentum material points relative to the center of O.

$$\frac{d\vec{K}_O}{dt} = \vec{M}_O$$

- the time derivative of the moment of momentum of the mater. points relative to any center is equal to the moment of force applied to the point, relative to the same center. Projecting vector equality on the coordinate axis. we get three scalar equations:

$$\frac{dK_x}{dt} = M_x, \text{ etc.}$$

- the derivative of the moment of the amount of motion of a material point relative to any axis is equal to the moment of the force applied to the point, relative to the same axis. Under the action of the central force passing through O,

$$M_O = 0, \Rightarrow \vec{K}_O = \vec{r} \times m\vec{v} = \text{const. } \vec{K}_O = 2m\vec{q} = \text{const, where } \vec{q} = \frac{1}{2}(\vec{r} \times \vec{v})$$

is the sector speed. Under the action of the central force, the point moves along a flat curve with a constant sectoral velocity, i.e. the radius vector of a point describes ("sweeps") equal areas at any equal time intervals (the law of areas) This law takes place during the movement of planets and satellites - one of Kepler's laws.

3.3.3 Force work. Power

Elementary work

$$dA = F_\tau ds, \ F_\tau$$

is the projection of the force on the tangent to the trajectory, directed in the direction of movement, or $dA = Fds\cos\alpha$.

If α is sharp, then $dA>0$, obtuse - <0, $\alpha=90^o$: $dA=0$.

$$dA = \overline{F} \cdot d\overline{r}$$

is the scalar product of the force vector and the elementary displacement vector of the point of its application;

$$dA = F_x dx + F_y dy + F_z dz$$

– analytical expression of the elementary work of the force. The work of the force on any finite displacement M_0M_1:

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} F_\tau ds .$$

If the force is constant, then

$$A_{(M_0M_1)} = F_\tau \cdot s = F \cdot s \cdot \cos\alpha .$$

Units of work: [1 J (joule) = 1 Nm].

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (F_x x + F_y y + F_z z) ds ,$$

because $dx = \dot{x}\, dt$, etc., then

$$A_{(M_0M_1)} = \int_{(t_0)}^{(t_1)} (F_x \dot{x} + F_y \dot{y} + F_z \dot{z}) dt .$$

The work force theorem: The work of the resultant force is equal to the algebraic sum of the work of the component forces on the same displacement $A = A_1 + A_2 + \ldots + A_n$.

Work of gravity:

$$A_{(M_0M_1)} = \pm P \cdot h ,$$

>0 if the start point is higher than the end point.

The work of the elastic force:

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} (-cx) dx = \frac{c}{2}(x_0^2 - x_1^2) = \frac{c}{2}[(\Delta\ell_{\text{нач}})^2 - (\Delta\ell_{\text{кон}})^2]$$

- the work of the elastic force is equal to half the product of the stiffness coefficient and the difference in the squares of the initial and final elongations (or compressions) of the spring.

The work of the friction force: if the friction force is const, then

$$A_{(M_0M_1)} = -F_{\text{тр}}s$$

it is always negative,

$$F_{\text{тр}}=fN,$$

f is the friction coefficient, N is the normal reaction of the surface.

The work of the gravitational force. Force of attraction (gravitation):

$$F = k\frac{m}{r^2}\text{, from } mg = k\frac{m}{r^2},$$

we find the coefficient. $k=gR^2$.

$$A_{(M_0M_1)} = km\int_{(M_0)}^{(M_1)}\frac{dr}{r^2} = mgR^2\left(\frac{1}{r_1}-\frac{1}{r_0}\right)$$

- does not depend on the trajectory.

Power is a value that determines the work per unit of time,

$$N = \frac{dA}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v} = F_x\dot{x} + F_y\dot{y} + F_z\dot{z}\,.$$

If the change in work occurs uniformly, then the power is constant: N=A/t. [1 W (watt) = 1 J / s, 1 kW (kilowatt) = 1000 W, 1hp (horsepower) = 75 kgfm / s = 736 W].

3.3.4 Theorem on the change in the kinetic energy of a point

In differential form:

$$d\left(\frac{mv^2}{2}\right) = \sum dA_k$$

- the total differential of the kinetic energy of a material point is equal to the elementary work of all forces acting on the point.

$$T = \frac{mv^2}{2}$$

is the kinetic energy of the material point. In the final form:

$$\frac{mv_2^2}{2} - \frac{mv_1^2}{2} = A_{1,2}$$

- the change in the kinetic energy of a material point, when it passes from the initial to the final (current) position, is equal to the sum of the work on this movement of all forces applied to the point.

A force field is an area, at each point of which a force acts on a material point placed in it, uniquely determined in magnitude and direction at any moment of time, i.e. should be known $\vec{F} = \vec{F}(\vec{r}, t)$. A non-stationary force field, if it $\vec{F}$ explicitly depends on t, a stationary force field, if the force does not depend on time. Stationary force fields are considered, when the force depends only on the position of the point: $\vec{F} = \vec{F}(\vec{r})$ and $F_x=F_x(x,y,z)$, etc. Properties of stationary force fields:

1) The work of the forces of a stationary field generally depends on the initial M_1 and final M_2 positions and trajectories, but does not depend on the law of motion of the mater. points.

2) The equality $A_{2,1} = - A_{1,2}$ takes place. For non-stationary fields, these properties are not satisfied.

Examples: gravity field, electrostatic field, elastic force field.

Stationary force fields, the work of the forces of which does not depend on the trajectory (path) of the movement of the mater. points and is determined only by its initial and final positions are called potential (conservative). $A_{1,2}^{I} = A_{1,2}^{II} = A_{1,2}$, where I and II are any paths, $A_{1,2}$ is the total value of the work. In potential force fields, there is such a function that uniquely depends on the coordinates of the points of the system, through which the projections of the force on the coordinate axes at each point of the field are expressed as follows:

$$X_i = \frac{\partial U}{\partial x_i}; Y_i = \frac{\partial U}{\partial y_i}; Z_i = \frac{\partial U}{\partial z_i}$$

The function U= U(x_1,y_1,z_1,x_2,y_2,z_2,…x_n,y_n,z_n) is called the strength function. Elementary work of field forces: $\delta A=\sum\delta A_i= dU$. If the force field is potential, the elementary work of forces in this field is equal to the total differential of the force function. The work of forces on the final displacement

$$A_{1,2} = \int_{(1)}^{(2)} dU = U_2 - U_1 ,$$

i.e. the work of forces in the potential field is equal to the difference between the values of the force function in the final and initial positions and does not depend on the shape of the trajectory. On a closed displacement, the work is equal to 0.

The potential energy P is equal to the sum of the work of the forces of the potential field on moving the system from a given position to zero. In the zero position P_0= 0. P=P(x_1,y_1,z_1,x_2,y_2,z_2,…x_n,y_n,z_n).

The work of the field forces on moving the system from the 1st position to the 2nd is equal to the difference in potential energies $A_{1,2}$= P_1–P_2. Equipotential surfaces are surfaces of equal potential. The force is directed along the normal to the equipotential surface.

The potential energy of the system differs from the force function, taken with a minus sign, by a constant value U_0: $A_{1,0} = P = U_0 - U$. Potential energy of the gravity field: P = mgz.

Potential energy field of central forces. The central force is a force that at any point in space is directed along a straight line passing through a certain point (center), and its modulus depends only on the distance r of a point of mass m to the center:

$$F = k\frac{m}{r^2}, \quad P = k\cdot m/r.$$

The central force is the gravitational force,

$$F = f\frac{m_1 m_2}{r^2}, \qquad P = f\cdot m_1\cdot m_2/r, \qquad f = 6.67\cdot 10\text{-}11 m^3/(kgf^2)$$

f – gravity constant. The first cosmic velocity

$$v1 = \sqrt{gR} \approx 7.9 \text{ km/s},$$

$R = 6.37 \cdot 10^6$ m is the radius of the Earth; the body enters a circular orbit. The second cosmic velocity:

$$v_{11} = \sqrt{2gR} \approx 11.2 \text{ km/s},$$

the trajectory of the body is a parabola, with $v > v_{11}$ it is a hyperbola.

3.3.5 Potential energy

Potencial restoring force energy of springs:

$$\Pi = \frac{c\lambda^2}{2},$$

λ – modulus of spring length increment. The work of the restoring force of the spring:

$$A_{1,2} = \frac{c\lambda_1^2}{2} - \frac{c\lambda_2^2}{2},$$

λ_1 and λ_2 are deformations corresponding to the start and end points of the path.

3.4 Dynamics of the material system

3.4.1 Center of gravity

A material system is a set of material points, the movement of which is interconnected. The mass of the system equal the sum of the masses of all points (or bodies) that form the system: $M = \sum m_k$. The center of mass (center of inertia) is a geometric point, the radius vector of which is determined by the equality:

$$\vec{r}_c = \frac{\sum m_k \vec{r}_k}{M},$$

where are $\vec{r}_k$ the radius vectors of the points that form the system. Center of mass coordinates:

$$x_c = \frac{\sum m_k x_k}{M}, \text{ etc.}$$

External forces F^e are the forces acting on the points of the system from the bodies that are not included in the system. Internal forces F^i are forces caused by the interaction of points included in the system. Properties of internal forces:

1) The geometric sum (principal vector) of all internal forces equal 0;

2) The geometric sum of the moments of all internal forces relative to an arbitrary point equal 0. Difference equations for the movement of a system of mother points:

$$m_k \frac{d^2\vec{r}_k}{dt^2} = \vec{F}_k^e + \vec{F}_k^i$$

or in projections on the coordinate axes:

$$m_k \ddot{x}_k = X_k^e + X_k^i,$$

etc. for each point (body) of the system.

3.4.2 Mass geometry

The moment of inertia of a mother point about a certain axis is the product of the mass m of this point and the square of its distance h to the axis: mh^2.

The moment of inertia of the body (system) relative to the Oz axis:

$$J_z = \sum m_k h_k^2.$$

With a continuous distribution of masses (body), the sum goes into an integral:

$$J_x = \int(y^2+z^2)dm;\ \ J_y = \int(z^2+x^2)dm;\ \ J_z = \int(x^2+y^2)dm$$

– relative to the coordinate axes.

$$J_z = M\cdot\rho^2,$$

ρ – radius of gyration of the body – distance from the axis to the point where the whole body must be concentrated so that its moment of inertia is equal to the moment of inertia of the body. The moment of inertia about the axis (axial moment of inertia) is always >0. Polar moment of inertia

$$J_o = \int(x^2+y^2+z^2)dm;\ \ \ J_x+J_y+J_z = 2J_o.$$

The centrifugal moment of inertia J_{xy} for the mother point is the product of its coordinates x and y and its mass m. For a body, the centrifugal moments of inertia are the quantities defined by the equalities:

$$J_{xy}=\int xy\ dm;\ J_{yz}=\int yz\ dm;\ J_{zx}=\int zx\ dm.$$

The centrifugal moments of inertia are symmetrical with respect to their indices, i.e. $J_{xy}=J_{yx}$ etc. Unlike axial, centrifugal moments of inertia can have any sign and vanish. The main axis of inertia of the body is called. an axis for which both centrifugal moments of inertia containing the index of this axis are equal to zero. For example, if $J_{xz}=J_{yz}=0$, then the z-axis is the main axis of inertia. The main central axis of inertia is called. the main axis of inertia passing through the center of mass of the body. 1) If the body has a plane of symmetry, then any axis perpendicular to this plane will be the main axis of inertia of the body for the point at which the axis intersects the plane. 2) If the body has an axis of symmetry, then this axis is the main axis of inertia of the body (axis of dynamic symmetry). Unit of all moments of inertia [kg m^2]

The centrifugal moment of inertia depends not only on the direction of the coordinate axes, but also on the choice of the origin.

Inertia tensor at a given point:

$$J = \begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{yx} & J_y & -J_{yz} \\ -J_{zx} & -J_{zy} & J_z \end{pmatrix}$$

Moments of inertia of some homogeneous bodies:
rod of mass m and length L:

$$J_y = \frac{mL^2}{12}; \; J_{y_1} = \frac{mL^2}{3}.$$

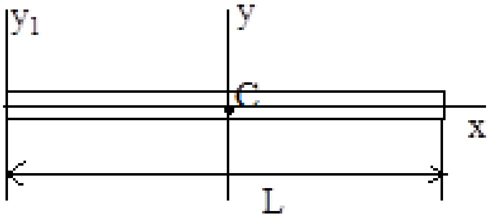


Figure 26

Homogeneous solid disk centered at point C with radius R and mass m:

$$J_{Cz} = \frac{mR^2}{2}.$$

Hollow cylinder:

$$J_{Cz} = \frac{m(R_1^2 + R_2^2)}{2},$$

cylinder with mass distributed over the rim (hoop):

$$J_{Cz} = mR^2.$$

The Huygens-Steiner theorem the moment of inertia of a body about an arbitrary axis is equal to the moment of inertia about an axis parallel to it and passing through the center of mass of the body plus the product of the body mass times the square of the distance between the axes:

$$J_{Oz'} = J_{Cz} + md^2 \ .$$

The smallest moment of inertia will be relative to the axis that passes through the center of mass. Moment of inertia about an arbitrary axis L:

$$J = J_x\cos^2\alpha + J_y\cos^2\beta + J_z\cos^2\gamma - 2J_{xy}\cos\alpha\cos\beta - 2J_{yz}\cos\beta\cos\gamma - 2J_{zx}\cos\gamma\cos\alpha,$$

if the coordinate axes are principal relative to their origin, then:

$$J = J_x\cos^2\alpha + J_y\cos^2\beta + J_z\cos^2\gamma.$$

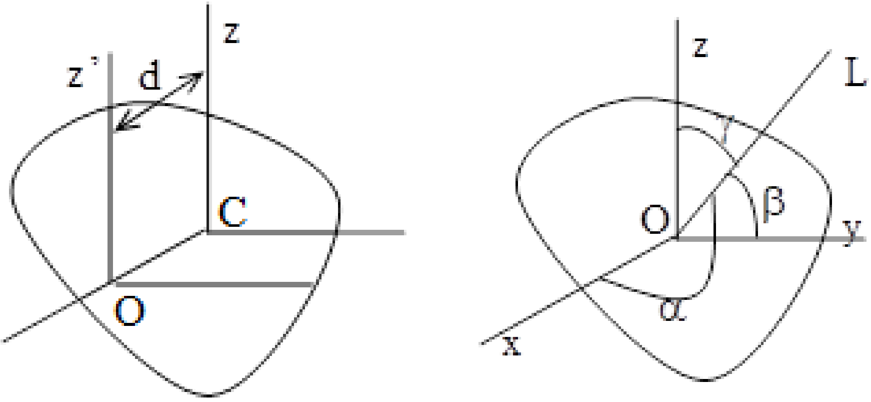


Figure 27

3.4.3 The theorem on the motion of the center of mass of the system

The product of the mass of the system and the acceleration of its center of mass is equal to the geometric sum of all external forces acting on the system

$$m\vec{a}_C = \sum \vec{F}_k^e$$

- the differential equation for the movement of the center of mass. In projections on the coordinate axes:

$$m\ddot{x}_C = \sum F_{kx}^e \text{ , etc.}$$

Law of conservation of motion of the center of mass. If the main vector (vector sum) of external forces remains equal to zero all the time, then the center of mass of the mechanical system is at rest or moves in a straight line and uniformly. Similarly, in projections on the axis, if $\sum F^e_{kx} = 0 \;\Rightarrow\; \ddot{x}_C = \dot{v}_{Cx} = \text{const}$, if at the same time at the initial moment $v_{Cx0}= 0$, то $\Rightarrow \dot{x}_C = 0 \Rightarrow x_C=$ const.

The amount of motion of the system Q (sometimes denoted by K) is a vector equal to the geometric sum (main vector) of the amounts of motion of all points of the system:

$$\vec{Q} = \sum m_k \vec{v}_k = M\vec{v}_C,$$

M is the mass of the entire system, v_C is the velocity of the center of mass.

3.4.4 The theorem on the change in the momentum of a system

$$\frac{d\vec{Q}}{dt} = \vec{F}^e$$

The time derivative of the momentum of a mechanical system is geometrically equal to the main vector of external forces acting on this system. In projections:

$$\frac{dQ_x}{dt} = F^e_{kx}, \text{ etc.}$$

The theorem on the change in the number of motion of the system in integral form:

$$\vec{Q}_1 - \vec{Q}_0 = \sum \int_0^{t_1} \vec{F}^e_k dt,$$

where $\sum \int_0^{t_1} \vec{F}^e_k dt = \sum \vec{S}^e_k$ are the impulses of external forces.

In projections:

$$Q_{1x} – Q_{0x} = \sum S^e_{kx}, \text{ etc.}$$

the amount of motion of the system for a certain period of time is equal to the sum of the impulses acting on the system of external forces for the same period of time.

The law of conservation of momentum - if the sum of all external forces acting on the system equal 0, then the momentum vector of the system will be constant in magnitude and direction:

$$\sum \vec{F}_k^e = 0 \Rightarrow \vec{Q} = \text{const},$$

similarly in projections:

$$\sum F_{kx}^e = 0 \Rightarrow Q_x = \text{const, etc.}$$

It follows from the law that internal forces cannot change the total amount of motion of the system. A body of variable mass, whose mass is constantly changing over time m = f (t) (examle: a rocket whose fuel is decreasing). The differential equation of motion of a point of variable mass:

$$m\frac{dV}{dt} = \vec{F}^e + \vec{u}\frac{dm}{dt}$$

is the Meshchersky equation, u is the relative velocity of the separated particles.

$$\vec{\Phi} = \vec{u}\frac{dm}{dt}$$

- reactive force,

$$\frac{dm}{dt} = -G_{\text{сек}}, \quad \vec{\Phi} = -\vec{u} \cdot G_{\text{сек}}$$

- second fuel consumption. The reactive force is directed in the opposite direction of the relative speed of the fuel outflow.

Tsiolkovsky's formula:

$$v_1 = v_0 + u \cdot \ln(1 + \frac{m_\text{т}}{m_k})$$

- determines the speed of the rocket, when all the fuel is used up - the speed at the end of the active section, m_t is the mass of the fuel, m_k is the mass of the rocket body, v_0 is the initial speed.

$$z = \frac{m_0}{m_k}$$

is the Tsiolkovsky number, m_0 is the launch mass of the rocket. From the operating mode of the rocket engine, i.e. the speed of the rocket at the end of the burning period does not depend on how quickly the fuel is burned. To achieve the first space velocity of 7.9 km/s, at $m_0/m_k = 4$, the ejection velocity must be 6 km/s, which is difficult to achieve, therefore composite (multi-stage) rockets are used.

The main moment of the quantities of motion material system (kinetic moment) $\vec{K}_o$ - a value equal to the geometric sum of the moments of the quantities of motion of all points of the system relative to the center O.

$$\vec{K}_o = \sum \vec{r}_i \times m_i \vec{v}_i \, .$$

3.4.5 Theorem on the change in the kinetic moment

The theorem on the change in the momentum of the quantities of motion of the system (theorem on the change in the kinetic moment):

$$\frac{d\vec{K}_o}{dt} = \sum \vec{M}_{io}^E = \vec{M}_o^E$$

is the time derivative of the mechanical momentum. system relative to some fixed center is geometrically equal to the main moment of external forces acting on this system relative to the same center. Similar equalities with respect to the coordinate axes:

$$\frac{dK_x}{dt} = M_x^E \text{, etc.}$$

Law of conservation of momentum: if $\vec{M}_o^E = 0$, then $\vec{K}_o = \text{const}$. The main moment of the momentum of the system is a characteristic of the rotational motion. The angular momentum of a rotating body relative to the axis of rotation is equal to the product of the moment of inertia of the body relative to this axis and the angular velocity of the body:

$$K_z = J_z\omega.$$

If $M_z = 0$, then $J_z\omega = \text{const}$, J_z is the moment of inertia of the body.

3.4.6 The kinetic energy of the system

The kinetic energy of the system is a scalar value T, equal to the arithmetic sum of the kinetic energies of all points in the system:

$$T = \sum \frac{m_k v_k^2}{2}.$$

If the system consists of several bodies, then $T = \sum T_к$. Forward movement:

$$T_{FORWARD} = \frac{1}{2} m v^2.$$

Rotational motion:

$$T_{ROTATIONAL} = \frac{1}{2} J_z \omega^2,$$

J_z – moment of inertia relative to the axis of rotation.

Planar-parallel (flat) motion:

$$T_{PLANAR} = \frac{1}{2} m v_C^2 + \frac{1}{2} J_C \omega^2,$$

v_C – velocity of the center of mass. General case:

$$T = \frac{1}{2} m v_C^2 + \frac{1}{2} J_{CP} \omega^2,$$

J_{CP} is the moment of inertia of the body relative to the instantaneous axis.

Koenig's theorem:

$$T=\frac{1}{2}mv_C^2+\sum\frac{m_i v_{ir}^2}{2}$$

- the kinetic energy of a mechanical system is equal to the sum of the kinetic energy of the center of mass of the system, the mass of which is equal to the mass of the entire system, and the kinetic energy of this system in its relative motion relative to the center of mass.

Work of force:

$$A_{(M_0M_1)} = \int_{(M_0)}^{(M_1)} F_\tau ds,$$

work of moment:

$$A = \int_0^{\varphi_1} Md\varphi.$$

Power:

$$N= Fv, \qquad N= M_z\omega.$$

The theorem on the change in the kinetic energy of the system: in differential form:

$$dT = \sum dA_k^e + \sum dA_k^i, \quad dA_k^e, \; dA_k^i$$

– elementary work of external and internal forces acting on the point, in the final form:

$$T_2 - T_1= \sum A_k^e + \sum A_k^i.$$

For an immutable system

$$\sum A_k^i = 0 \text{ and } T_2 - T_1 = \sum A_k^e,$$

i.e. the change in the kinetic energy of a rigid body at some displacement is equal to the sum of the work of external forces acting on the body at this displacement. If the sum of the work of the reactions of the bonds on any possible displacement of the system is equal to zero, then such bonds are called ideal. Efficiency factor (efficiency):

$$\eta = \frac{A_{res.forces}}{A_{w\exp}} < 1,$$

$A_{res.forces.}$- the work of useful resistance forces (forces for which the machine is intended),

$$A_{w\ exp} = A_{res.forces.} + A_{harm.forces.}$$

- work expended, $A_{harm.forces.}$ - work of harmful resistance forces (friction forces, air resistance, etc.).

$$\eta = N_{mach}/N_{dv},\ N_{mach}$$

is the useful power of the machine, N_{dv} is the power of the engine that sets it in motion.

The law of conservation of total mechanical energy:

$$T + P = \text{const.}$$

If the system moves under the action of potential forces, then the sum of the kinetic and potential energies remains constant. (T + P is the energy integral). Potential forces - forces whose work does not depend on the type of trajectory along which the point moves (gravity, elasticity force) Non-potential - for example: friction forces. Mechanical energy is the sum of kinetic and potential energies. The consumption of mechanical energy usually means its transformation into heat, electricity, sound or light, and the influx of mechanical energy is associated with the reverse process of converting various types of energy into mechanical energy.

3.5 Rigid Body Dynamics

3.5.1 Differential equations of translational motion of a rigid body

$$m\ddot{x}_C = \sum X_i^e ,$$

etc. X_i^e is the projection of an external force. All points of the body move in the same way as its center of mass C. To carry out translational motion, it is necessary that the main moment of all external forces relative to the center of mass be equal to 0: $\vec{M}_c^e$=0.

Differential equations of rotation of a rigid body around a fixed axis:

$$J_z\ddot{\varphi} = M_z^e ,$$

J_z is the moment of inertia of the body about the axis of rotation z, M_z^e is the moment of external forces about the axis of rotation (torque).

$$J_z\varepsilon = M_z^e ,\ \varepsilon$$

- angular acceleration, the greater the moment of inertia for a given M_z^e, the lower the acceleration, i.e. the moment of inertia during rotational motion is an analogue of mass during translational. Knowing M_z^e, you can find the law of rotation of the body φ=f(t), and, conversely, knowing φ=f(t), you can find the moment. Special cases:

1) if M_z^e = 0, then ω = const – the body rotates uniformly;

2) M_z^e = const, then ε = const – rotation is equally variable.

The equation is similar to the differential equation of rectilinear motion of a point

$$m\ddot{x} = F_x .$$

3.5.2 Differential equations of body rotation around a fixed axis

A physical pendulum is a rigid body that oscillates around a fixed horizontal axis under the influence of gravity. Rotational motion equation:

$$J_o \frac{d^2\varphi}{dt^2} = -Pa\sin\varphi,$$

denoting $\frac{Pa}{J_o} = k^2$, we obtain the differential equation of pendulum oscillations:

$$\ddot{\varphi} + k^2 \sin\varphi = 0,$$

k is the frequency of pendulum oscillations. Considering small oscillations, we can consider $\sin\varphi \approx \varphi$, then

$$\ddot{\varphi} + k^2\varphi = 0$$

- the differential equation of harmonic oscillations. Solution of this equation:

$$\varphi = C_1\cos kt + C_2 \sin kt \quad \text{or} \quad \varphi = \alpha\sin(kt + \beta),$$

α – amplitude of pendulum oscillations, β – initial phase of oscillations. The period of small oscillations of the physical pendulum is

$$T = 2\pi/k = 2\pi\sqrt{J_o/(Pa)}.$$

For small oscillations of the pendulum, the period does not depend on the angle of the initial deflection, this result is approximate. For a mathematical pendulum (a material point suspended on an inextensible thread and moving under the action of gravity), we have differential equations of motion:

$$\ddot{\varphi} + \frac{g}{L}\sin\varphi = 0,$$

L is the length of the thread. If

$$L = \frac{J_o}{ma},$$

then the mathematical pendulum will move in the same way as the physical one (the oscillation period is the same). The value L is called the reduced length of the physical pendulum. The point K, which is located at a distance OK=L from the axis of the suspension, is called the swing center of the physical pendulum. If the suspension axis is taken at point K, then point O will be the center of swings and vice versa - the property of reciprocity. The distance OK is always greater than OC, i.e. the center of oscillation is always located below the center of mass.

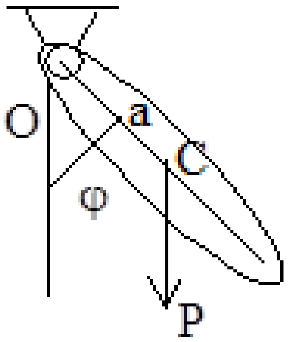


Figure 28

3.5.3 Dynamics of plane motion of a rigid body

The position of the body is determined by the position of the pole and the angle of rotation of the body around the pole. Differential equations of plane motion of a rigid body:

$$m\ddot{x}_C = \sum X_i^e \quad m\ddot{y}_C = \sum Y_i^e \ ; \ J_C\ddot{\varphi} = \sum m_C(F_i^e) \ ;$$

C – body mass center, J_C – body moment of inertia relative to the axis perpendicular to the body motion plane and passing through its mass center.

3.5.4 d'Alembert's principle (kinetostatics method)

At each moment of movement, the sum of active forces, reactions of connections and inertial forces is equal to zero

$$\sum(\vec{F}_k^e + \vec{F}_k^i + \vec{F}_k^{in}) = 0$$

- d'Alembert's principle for a material point.

$\vec{F}_k^e$ - external force, $\vec{F}_k^i$ - internal force. $F_k^{in} = -m_k w_k$ - inertia force, the sign (-) indicates that the inertia force is directed in the opposite direction to acceleration.

For the system, the equation of moments is added:

$$\sum[\vec{m}_o(\vec{F}_k^e) + \vec{m}_o(\vec{F}_k^i) + \vec{m}_o(\vec{F}_k^{in})] = 0 .$$

Designate: $\vec{F}^{in} = \sum \vec{F}_k^{in}$ - the main vector of the forces of inertia, $\sum \vec{m}_o(\vec{F}_k^{in}) = \vec{M}_o^{in}$ - the main moment of the forces of inertia. Considering that the geometric sum of the internal forces and the sum of their moments is equal to zero $\sum \vec{F}_k^i = 0$, $\sum \vec{m}_o(\vec{F}_k^i) = 0$, we obtain:

$$\sum \vec{F}_k^e + \vec{R}^{in} = 0 , \ \sum \vec{m}_o(\vec{F}_k^e) + \vec{M}_o^{in} = 0$$

are the equations of kinetostatics. d'Alembert's principle for a system - if at any time to each point of the system, in addition to the actually acting forces, the corresponding inertia forces are applied, then the resulting system of forces will be in equilibrium and the equations of statics can be applied to it. This simplifies the problem solving process.

The main vector of inertial forces is equal to the product of the body mass and the acceleration of its center of mass and is directed opposite to this acceleration

$$\vec{F}^{in} = -\sum m_k \vec{w}_k = -M\vec{w}_c .$$

The main moment of the forces of inertia depends on the type of movement: with translational motion $\vec{M}_o^{in} = 0$; when flat $\vec{M}_c^{in} = -J_c \vec{\varepsilon}$, during rotation around the z axis passing through the center of mass of the body,

$$\vec{M}_c^{in} = -J_z \vec{\varepsilon} .$$

3.5.5 *Determination of reactions during the rotation of a rigid body around a fixed axis*

When a body rotates around a fixed axis, dynamic pressures arise on the supports. Their determination is conveniently solved by the kinetostatics method. We apply inertial forces for each point: centrifugal and rotational

$$F_i^c = m_i r_i \omega^2 , \qquad F_i^{rot} = m_i r_i \varepsilon ,$$

r_i is the distance from the point to the axis of rotation. Projecting the sum of these forces on the axis and taking into account that

$$\sum m_i x_i = m x_C \quad \text{and} \quad \sum m_i y_i = m y_C ,$$

C is the center of mass, we obtain the projections of the main vector of inertia forces:

$$F_x^{in} = m x_c \omega^2 + m y_c \varepsilon , \; F_y^{in} = m y_c \omega^2 - m x_c \varepsilon .$$

The projection of the main moment of the forces of inertia is equal to the sum of the moments of the centrifugal and rotational forces of inertia about the coordinate axes:

$$M_x^{in} = -\omega^2 \sum m_i y_i z_i + \varepsilon \sum m_i z_i x_i = -J_{yz} \omega^2 + J_{zx} \varepsilon ,$$
$$M_y^{in} = \omega^2 \sum m_i z_i x_i + \varepsilon \sum m_i y_i z_i = J_{zx} \omega^2 + J_{yz} \varepsilon ,$$
$$M_z^{in} = -\sum m_i r_i \varepsilon r_i = -\varepsilon \sum m_i r_i^2 = -J_z \varepsilon ,$$
$$J_{yz} = \sum m_i y_i z_i , \quad J_{zx} = \sum m_i z_i x_i , \quad J_z = \sum m_i r_i^2$$

are the centrifugal moments of inertia,

Taking into account external forces, we can write the equilibrium equations for kinetostatics:

$$\sum X_i^e + X_A + X_B + m x_C \omega^2 + m y_C \varepsilon = 0,$$
$$\sum Y_i^e + Y_A + Y_B + m y_C \omega^2 - m x_C \varepsilon = 0,$$

$$\sum Z_i^e + Z_A = 0,$$
$$\sum M_{ix}^e - Y_B h - J_{yz}\omega^2 + J_{zx}\varepsilon = 0,$$

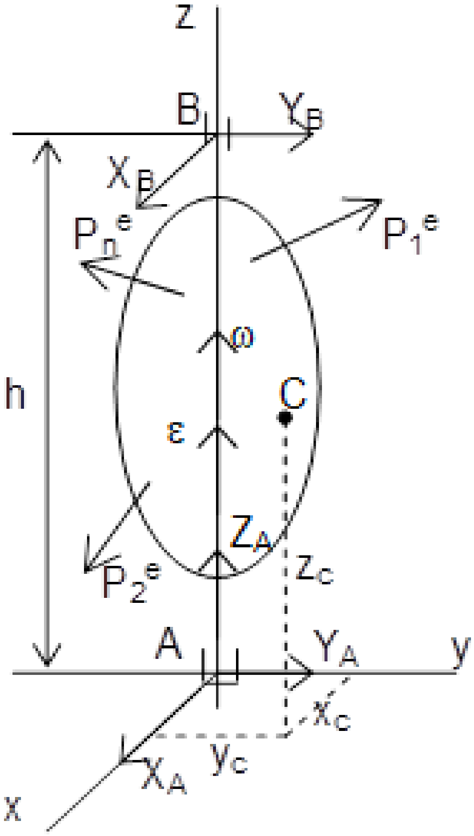


Figure 29

$$\sum M_{iy}^e + - X_B h + J_{zx}\omega^2 + J_{yz}\varepsilon = 0,$$
$$\sum M_{iz}^e - J_z\varepsilon = 0.$$

The last equation does not contain the reaction of the supports and is a differential equation for the rotation of the body. The remaining five equations allow us to determine five unknown reactions. The dynamic components of the reactions are determined by terms that depend on the forces of inertia.

Conditions for the absence of dynamic components:

$$m x_c\omega^2 + m y_c\varepsilon = 0, \qquad m y_c\omega^2 - m x_c\varepsilon = 0,$$
$$-J_{yz}\omega^2 + J_{zx}\varepsilon = 0, \qquad J_{zx}\omega^2 + J_{yz}\varepsilon = 0,$$

where $x_C = 0$, $y_C = 0$, $J_{yz} = 0$, $J_{zx} = 0$, which means that the center of gravity must be on the body's rotation axis, and the body's rotation axis z must be the main axis of inertia of the body. Those the axis of rotation must be the main central axis of inertia of the body (the axis passing through the center of mass of the body, and the centrifugal moments of inertia with the index of this axis are equal to zero). To fulfill this condition, a special balancing of rapidly rotating bodies is carried out.

3.6 Fundamentals of analytical mechanics

3.6.1 Principle of possible movements

Possible (virtual) displacements of the system (δs, $\delta\varphi$) – any set of infinitely small displacements of system points allowed at the moment by the constraints imposed on the system. Possible displacements are considered as quantities of the first order of smallness, while neglecting the values of higher orders of smallness. Those. curvilinear displacements of points are replaced by rectilinear segments laid tangentially to their trajectories.

The number of possible displacements of a system independent of each other is called the number of degrees of freedom of this system. For example: the ball on the plane can move in any direction, but any possible movement can be obtained as the geometric sum of two movements along two mutually perpendicular axes. A free rigid body has 6 degrees of freedom.

Possible (virtual) work δA is the elementary work that the force acting on the parent point could do on the possible displacement of this point.

The bonds are ideal if the sum of the elementary work of the reactions of these bonds for any possible displacement of the system is equal to zero, i.e. $\Sigma\delta A^r=0$.

3.6.2 General Equation of Dynamics

The principle of possible displacements: for the equilibrium of a mechanical system with ideal connections, it is necessary and sufficient that the sum of the elementary works of all active forces acting on it for any possible displacement be equal to zero.

$$\sum \vec{F}_k \cdot \delta \vec{r}_k = 0$$

or in projections:

$$\sum (F_{kx} \cdot \delta x_k + F_{ky} \cdot \delta y_k + F_{kz} \cdot \delta z_k) = 0 .$$

The principle of possible displacements gives in a general form the equilibrium conditions for any mechanical system, gives a general method for solving problems of statics.

If the system has several degrees of freedom, then the equation of the principle of possible displacements is made up for each of the independent displacements separately, i.e. there will be as many equations as the system has degrees of freedom.

The general equation of dynamics

$$\sum \delta A_k^a + \sum \delta A_k^{in} = 0$$

- when a system moves with ideal connections at any given moment in time, the sum of the elementary works of all applied active forces and all inertia forces on any possible movement of the system will be equal to zero. The equation uses the principle of possible displacements and the d'Alembert principle and allows one to compose differential equations of motion for any mechanical system. Gives a general method for solving problems of dynamics. The sequence of compilation: a) the given forces acting on it are applied to each body, and also the forces and moments of inertia force pairs are conditionally applied; b) inform the system of possible movements; c) compose the equations of the principle of possible displacements, considering the system to be in equilibrium.

3.6.3 Lagrange equations of the 2nd kind

Lagrange equations of the 2nd kind:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i , \text{ (i=1,2…s)}$$

– second-order differential equations,

s – number of degrees of freedom of the system (number of independent coordinates);

q_i – generalized coordinate (displacement, angle, area, etc.);

$\dot{q}_i$ – generalized speed (linear speed, angular, sectoral, etc.),

$T = T(q_1, q_2, \ldots, q_S, \dot{q}_1, \dot{q}_2 \ldots \dot{q}_s, t)$ is the kinetic energy of the system, Q_i is the generalized force (force, moment, etc.), its dimension depends on the dimension of the generalized coordinate and the dimension of the work.

To calculate the generalized force, for example, Q_1, we set a possible displacement for which all variations of the generalized coordinates, except for δq_1, are equal to zero:

$$\delta q_1 \neq 0,\ \delta q_2 = \delta q_3 = \ldots = \delta q_S = 0.$$

We calculate the possible work δA_1 of all active forces applied to the system on this displacement. Having

$$\delta A_1 = Q_1 \delta q_1$$

we find

$$Q_1 = \frac{\delta A_1}{\delta q_1}.$$

If the forces acting on the system are potential (conservative) (for example, gravity, elastic forces), then

$$Q_i = -\frac{\partial \Pi}{\partial q_i},$$

$\Pi = \Pi(q_1, q_2, \ldots, q_S, t)$ is the potential energy.

The Lagrange function is introduced: $L = T - \Pi$, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

the Lagrange equations of the second kind for a conservative system.

With stationary links (links that do not depend on time), t is not included in the expression for kinetic energy, then

$$T = \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{s} a_{ij}\dot{q}_i\dot{q}_j$$

is the quadratic form of the generalized velocities, $a_{ij} = a_{ji}$ are the coefficients of inertia. The quadratic form is always positive.

3.7 Foundations of the gyroscope theory

3.7.1 Basic definitions and concepts

In accordance with the current state and prospects for the development of gyroscopic technology, gyroscopes in a broad sense are called a device containing a rotating or oscillating element and allowing, on this basis, to detect and measure rotation in the inertial space of the base on which this device is installed. This definition corresponds to the very meaning of the term gyroscope, introduced in 1852 by the french physicist Jean Bernard Léon Foucault (1819-1868), formed from two greek words: rotation and observe. That is, in a free translation, a gyroscope is a rotation indicator.

As a gyroscope, rotating solid, liquid and gaseous bodies can be used, the possibility of using the gyroscopic properties of particles - atomic nuclei or electrons with spin or orbital moments - has been practically proven. Laser gyroscopes have been created on the basis of optical quantum generators.

However, at present, in technical devices, especially in the navy, gyroscopes are most widely used, which use a dynamically symmetric rapidly rotating solid body (rotor), suspended in such a way that its own rotation axis can arbitrarily change direction in space. Therefore, the main parts of the gyroscope are the rotor and its suspension.

The axis of the rotor's own rotation is called the main axis of the gyroscope (the axis of the figure). Any two other axes lying in the plane of the rotor's own rotation and perpendicular to each other and to the main axis are called equatorial.

The concept of "high-speed rotor" means that the angular velocity of the rotor's own rotation is many orders of magnitude greater than the angular velocities that it can have relative to the equatorial axes,

The center of the gyroscope's suspension is that point which remains the only fixed point during all rotational movements of the rotor. If the center of mass of the gyroscope coincides with the center of the suspension, then the gyroscope is called astatic, or balanced, if it does not match, it is heavy.

A gyroscope is said to be free if no moment of external forces acts on it. In engineering, a free gyroscope is often understood as an astatic gyroscope with extremely small moments of friction forces on the suspension.

Consider the suspensions used in gyroscopes. The degree of perfection of a gyroscope built on the basis of a solid rotor largely depends on the quality of its suspension. Through the suspension, the gyroscope rotor is connected to the base (object, platform) on which it is installed. The suspension of the gyroscope is considered to be the better, the less the angular movements of the base are transmitted to the rotor.

All gyroscopes (gyroscopic sensing elements) can be divided into two classes depending on what is the suspension object:

a chamber (shell) containing a rapidly rotating rotor (or a system of rotors). In this class of gyroscopes, a gimbal, hydrostatic (in combination with an electromagnetic or elastic suspension), as well as a gas-static suspension are used;

the fast rotating rotor. In this class of gyroscopes, suspensions are used - electrostatic, hydrodynamic, electromagnetic, cryogenic, gas-dynamic, and also elastic rotating.

In those gyroscopes in which an electrostatic or electromagnetic field or liquid or gas pressure forces are used for suspension, the rotor itself or the chamber containing the rotor, as a rule, has a spherical shape. This form is the most convenient from the point of view of ensuring the symmetry of the acting support forces.

If the fundamentally necessary components of a gyroscope are a rotor and a suspension, then a gyroscope intended for use in a gyroscopic device must have: a rotor (a chamber with a rotor), a drive (to give the rotor its own rotational motion), and in some cases a sensor angle (for tracking the angular position of the gyroscope), and a moment sensor for imposing control and corrective moments.

Consider the properties of a gyroscope with three degrees of freedom. The simplest version of a gyroscope with three degrees of freedom is a gyroscope in a gimbal suspension (fig. 30).

Rotor 1 is suspended in a system of rings so that it can rotate around the X-X axis relative to the inner ring 2 (natural rotation), the inner ring around the Y-Y axis relative to the outer ring J. and the latter around the Z-Z axis relative to grounds. The point O of the intersection of the axes X - X, Y - Y and Z - Z is the center of the gyroscope's suspension.

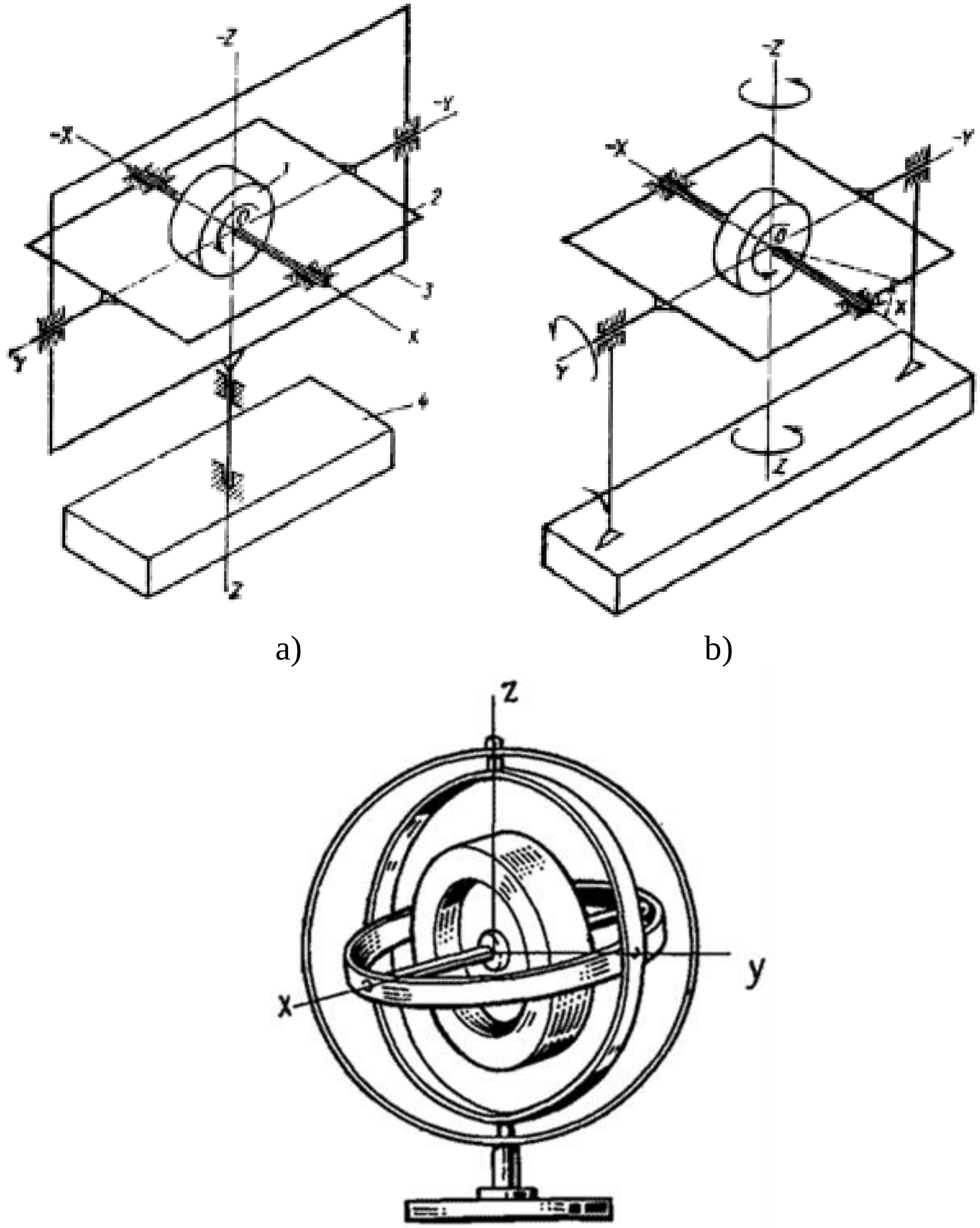
The first property is as follows. The main axis of a free gyroscope tends to keep its direction unchanged in inertial space. This means that if the main axis is directed to any star, then with any movement of the base on which the gyroscope is installed, it will invariably point to this star, changing its orientation with respect to the coordinate system associated with the Earth. This property was first used by L. Foucault to prove the daily rotation of the Earth.

The second property is that under the action of an external force applied to the inner or outer ring and creating a moment that does not coincide in direction with the main axis of the gyroscope, the latter will move not in the direction of the force (as it would be with a non-rotating rotor), but perpendicular to this direction. A similar property of a gyroscope is called precession. The precessional motion occurs at a constant angular velocity, i.e. is inertialess.

The third property is expressed as follows. Under the action of a force impulse (impact), the main axis of the gyroscope practically does not change its original direction, but only makes rapid oscillations around the equilibrium position. These fluctuations are called nutation. They are especially noticeable at a low angular velocity of the rotor's own rotation.

The properties of a gyroscope with three degrees of freedom are used in such devices as gyrocompasses, gyrohorizons, indicator-type gyro-stabilizers.

Property of a rotating symmetric body with two degrees of freedom. We get a gyroscope with two degrees of freedom if we deprive a gyroscope with three degrees of freedom of one degree of freedom, for example, around the Z-Z axis (fig. 30, a), by rigidly connecting ring 3 with base 4. As a result, we will have a device, shown in (fig. 30, b).



a) b)

c)

Figure 30. Gyroscope in cardan suspension:
a) with three degrees of freedom; b) with two degrees of freedom,
c) the appearance of a real gyroscope

A gyroscope with two degrees of freedom does not have any of the properties that a gyroscope with three degrees of freedom has. The property inherent only in a gyroscope with two degrees of freedom is as follows. If the base on which the gyroscope is installed is given rotation around the Z-Z axis, i.e., an axis that does not coincide with the axis of the rotor’s own rotation and the suspension axis Y-Y, then the rotor, together with the suspension ring, rotates around the suspension axis until the axis X - X of the rotor's own rotation coincides with the axis of rotation of the base, i.e. with the axis Z - Z of the forced rotation of the gyroscope.

Based on the property of a gyroscope with two degrees of freedom, gyrotachometers (differentiating gyroscopes), integrating gyroscopes, etc.

Property of a gyroscope with one degree of freedom. As long as the base on which the body with its own rotation is installed is motionless, the body does not have any gyroscopic properties. They arise if the rotation of the base causes a forced rotation of the body around an axis that does not coincide with the axis of its own rotation. In accordance with the property of a gyroscope with two degrees of freedom, the body's own rotation axis tends to coincide with the forced rotation axis. This movement is prevented by the supports (bearings) of the main axle. The action of the rotor on the supports is expressed as the application of forces to them, which are called gyroscopic forces.

It is necessary to take into account the gyroscopic effect of bodies having one degree of freedom in their own rotation when they are installed on objects that change the direction of movement (ship or aircraft turbine, helicopter propeller).

To prove the properties of a gyroscope, it is necessary to use the methods of analytical mechanics.

3.7.2 The momentum theorem

The momentum theorem is of great importance in the theory of gyroscopes. With its help, the main properties of the gyroscope are proved, which were previously considered only the sides of their external manifestation.

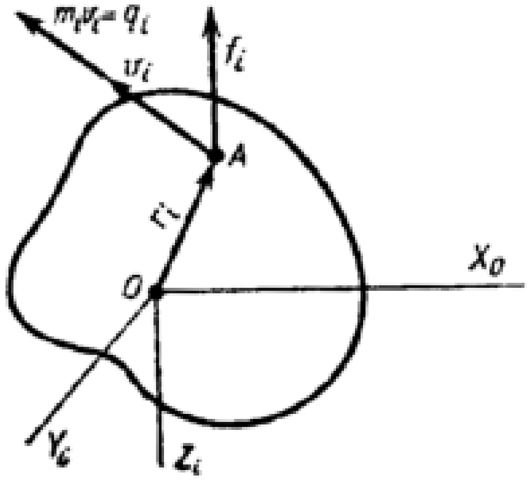


Figure 31

We will prove the theorem on the kinetic moment, keeping in mind an arbitrary rigid body whose point is fixed (fig. 31). Let us single out in the body a material point having a mass. The position of the point is completely determined by the radius vector. Suppose that an external force acts on the body at a specified point. The moment of this force relative to a fixed point is determined by the following vector product:

$$\bar{l}_i = \bar{r}_i \times \bar{f}_i$$

The value of the moment of force:

$$l_i = r_i \cdot f_i \cdot \sin(\bar{r}_i;\bar{f}_i)$$

Under the action of a force, the body comes into rotation and each material point of the body acquires a linear velocity. As you know, the product of the mass of a material point and its linear speed is called the momentum of a material point:

$$\bar{q}_i = m\bar{v}_i$$

The momentum vector has the same direction as the linear velocity vector. Just as the moment of force relative to the point O was determined above, the moment of any other vector can be determined. This moment is called the angular momentum of a material point with mass relative to point O:

$$\bar{h}_i = \bar{r}_i \times \bar{q}_i = \bar{r}_i \times m_i \bar{v}_i$$

Let's find the first time derivative of the angular momentum vector:

$$\frac{d\bar{h}_i}{dt} = \frac{d\bar{r}_i}{dt} \times m_i \bar{v}_i + \bar{r}_i \times m_i \frac{d\bar{v}_i}{dt} \qquad (3.1)$$

Since the first derivative of the radius vector with respect to time is the velocity vector of its end, the vector product can be represented as a product of collinear vectors.

Consequently, expression (1.1) will take a simpler form:

$$\frac{d\bar{h}_i}{dt} = \bar{r}_i \times m_i \frac{d\bar{v}_i}{dt} \qquad (3.2)$$

According to Newton's second law:

$$\bar{f}_i = m_i \bar{j}_i = m_i \frac{d\bar{v}_i}{dt} \qquad (3.3)$$

Substituting the value of formula (3.3) into formula (3.2), we obtain

$$\frac{d\bar{h}_i}{dt} = \bar{r}_i \times \bar{f}_i$$

Since the right side of this equality is the moment of force relative to the point O, then

$$\frac{d\bar{h}_i}{dt} = \bar{l}_i$$

Considering a rigid body as a collection of n material points and assuming that k external forces act on the body, summing up, we can write

$$\sum_{i=1}^{n}\frac{d\overline{h}_i}{dt}=\sum_{j=1}^{k}\overline{l}_j \qquad (3.4)$$

We write expression (3.4) in another form, where the vector H is the moment of momentum of the body relative to the point O or the kinetic moment, and the vector L is the main moment of external forces:

$$\frac{d\overline{H}}{dt}=\overline{L} \qquad (3.5)$$

The resulting formula (3.5) expresses the main theorem of the dynamics of a rigid body (theorem of the kinetic moment): *the first time derivative of the vector of the kinetic moment of the body is equal to the vector of the main moment of all external forces acting on the body.*

The derivative of the angular momentum must be understood in a vector sense. Then the derivative is the speed of the end of the vector, i.e.

$$\frac{d\overline{H}}{dt}=\overline{U} \qquad (3.6)$$

Comparing expressions (3.5) and (3.6), we obtain

$$\overline{U}=\overline{L} \qquad (1.7)$$

From here, the momentum theorem can be formulated as follows: *the linear velocity vector of the end of the angular momentum vector of a rigid body relative to some point is equal to the vector of the main moment of all forces acting on the body relative to the same point.* This formulation is known as Resal's theorem.

Let us consider in more detail what constitutes the kinetic moment H in relation to the gyroscope. Let's single out in the gyroscope, the rotor of which has the angular velocity of its own rotation, point A (fig.32). The moment of momentum of point A relative to point O of the center of suspension of the gyroscope:

$$\bar{h}_i = \bar{r}_i \times m_i \bar{v}_i$$

In this case

$$h_i = r_i m_i v_i \sin 90^\circ = r_i m_i v_i$$

Based on the relation

$$v_i = \Omega \cdot r_i$$

we get

$$h_i = m_i r_i^2 \Omega$$

When summing over all points of the gyroscope rotor, we find

$$H = \sum_{i=1}^{n} h_i = \sum_{i=1}^{n} m_i r_i^2 \Omega = \Omega \sum_{i=1}^{n} m_i r_i^2$$

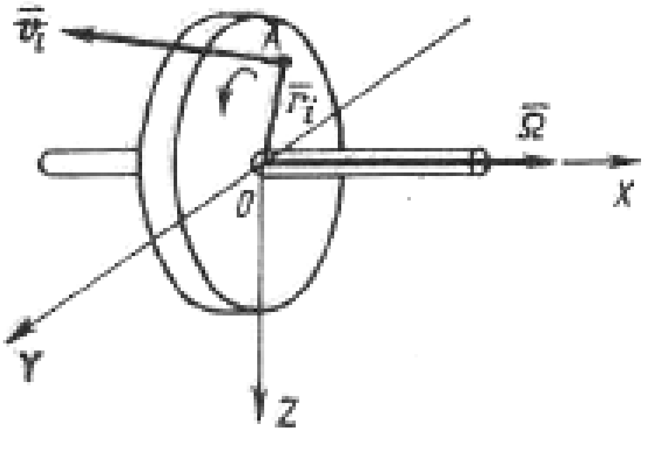


Figure 32

However, the sum in this expression is nothing but the moment of inertia of the gyroscope rotor relative to the X axis. Finally, we get

$$H = J_0 \Omega \qquad (3.8)$$

Thus, the own kinetic moment of the gyroscope is equal to the product of the axial moment of inertia of the gyroscope rotor and its own angular velocity.

The direction of the angular momentum vector H coincides with the direction of the angular velocity vector.

Suppose that at the time when the gyroscope rotates around its main axis X with an angular velocity , this axis does not remain stationary, but changes direction in space, rotating around the point O. How in this case to determine the angular momentum of the gyroscope with respect to the fixed point O?

Obviously, its value is not equal to the simple product of the moment of inertia and the angular velocity, and the direction does not coincide with the direction of the X axis, since the total vector of the kinetic moment is the resultant of the two components of the vectors of the kinetic moments. However, if the gyroscope rotates around the X-axis at a high angular velocity, while the X-axis changes its direction in space relatively slowly, when calculating the angular momentum, the movement of the X-axis can be neglected. In this case, the value of the angular momentum H is expressed by the formula (3.8), and the direction of the vector H coincides with the direction of the X axis.

The angular momentum H of a gyroscope is the most complete characteristic of a rotating body, since neither the mass of the body, nor its moment of inertia, nor the angular velocity of its rotation individually reflect its gyroscopic properties.

3.7.3 Applying the momentum theorem to proving the properties of a gyroscope

The momentum theorem makes it possible to prove the basic properties of a gyroscope.

Stability of the main axis of a free gyroscope. To prove this property, it is necessary to substitute the zero value L in expression (3.5), since the condition of a free gyroscope means that it is free from the action of any moments of external forces.

Therefore, for a free gyroscope:

$$d\overline{H}/dt = 0 \qquad \overline{H} = \mathrm{const}$$

In other words, the value and direction of the angular momentum of the gyroscope H are constant. Bearing in mind that in (3.8) (where the axial moment of inertia of the gyroscope is a constant), we can conclude that the angular velocity is also constant.

It is known that for a rapidly rotating gyroscope, the direction of the angular momentum H coincides with the direction of the X axis of the gyroscope, i.e., with the direction of its main axis. Consequently, the invariance of the direction H indicates the invariance of the direction in space of the main axis of the free gyroscope. This confirms the property of stability of the main axis of a free gyroscope in space.

Gyro precession. Consider a gyroscope (fig. 33), which is affected by a force F, creating a moment L around any axis that does not coincide with the main axis of the gyroscope (in the figure around the Y axis). As is known, the angular momentum H of a rapidly rotating gyroscope should be considered directed along the main axis. Denote the end of the vector H by N.

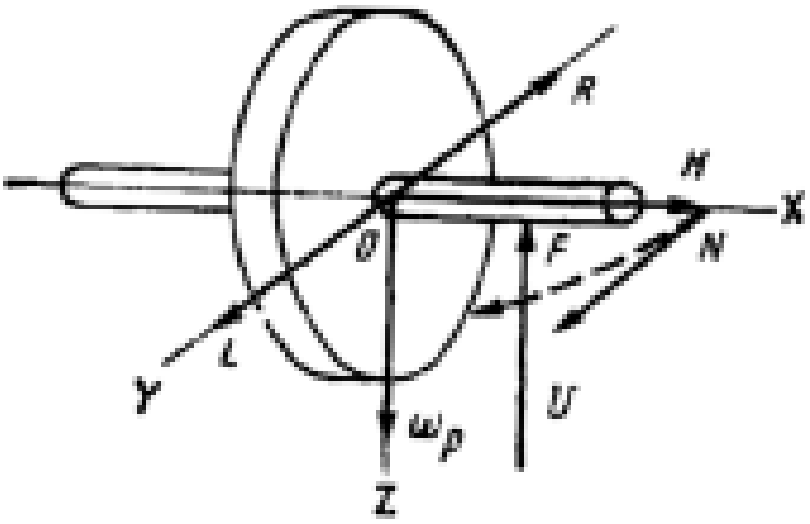


Figure 33

According to Resal's theorem, the expression U=L means that the end of the vector H, i.e. point N, acquires a linear velocity equal to and parallel to the moment vector L.

On fig. 33 it can be seen that the movement of the main axis of the gyroscope does not occur in the direction of the applied force F, but in the XOY plane perpendicular to the force in the direction of the moment of this force, i.e. we observe the case of the precessional movement of the gyroscope. It occurs when the moment L of the applied forces does not coincide in direction with the moment H.

Considering fig. 33 let us formulate a rule, using which, in practice, it is possible to determine the direction of precession: the precessional movement always takes place in the direction in which the angular momentum vector H turns in the shortest way to the moment vector L of the applied force. The angular velocity of the precessional motion of the gyroscope is numerically equal to the linear velocity U divided by the rotation radius OA, i.e. on H:

$$\omega_p = U/H$$

But by Resal's theorem U=L , so

$$\omega_p = L/H \qquad (3.9)$$

Formula (3.9) expresses the law of precession, which is very important in the applied theory of gyroscopes. Thus, the angular velocity of precession is directly proportional to the applied moment L of external forces and inversely proportional to the angular momentum H of the gyroscope.

Impact resistance. The mathematical expression (3.5) of the momentum theorem can be represented in finite increments

$$\frac{\Delta\overline{H}}{\Delta t} = \overline{L} \qquad \Delta\overline{H} = \overline{L}\Delta t \qquad (3.10)$$

The resulting expression can be interpreted as follows: the moment of the external force L, acting on the gyroscope for a small (in the limit of infinitely small) time, causes a small (in the limit of infinitely small) change in the kinetic moment H in direction (fig. 34).

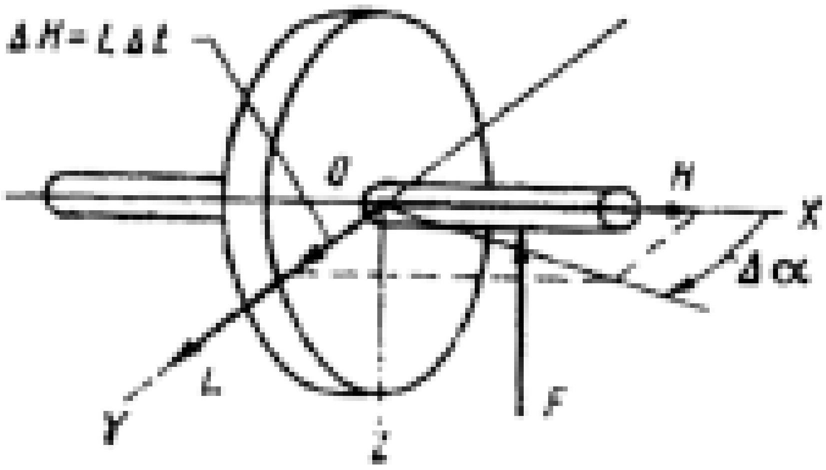


Figure 34

Angular change. If the moment of the external force acted for a very short time interval, then even with a large value of the moment L, the value of H changes insignificantly. Therefore, the angle of change is very small. This leads to an important practical conclusion that a force impulse (impact) can only slightly change the position of the main axis of a rapidly rotating gyroscope.

3.7.4 Gyroscopic moment

When studying the properties of precession, it was found that the main axis of the gyroscope does not move in the direction of the applied force, but perpendicular to the direction of the acting force. Such a phenomenon can occur only if a reaction force arises from the side of the gyroscope, balancing the external force applied to the gyroscope. This counterforce, which prevents the gyroscope from moving in the direction of the force, is called the gyroscopic reaction, and its moment external forces L) is called the moment of the gyroscopic reaction, or the gyroscopic moment.

From formula (3.9), expressing the law of precession, one can obtain the formula for the moment of the external force that causes the precession motion:

$$L = H \cdot \omega_p$$

This moment of the external force is balanced by the gyroscopic moment R, equal in value and opposite in direction to the moment L, i.e. R = -L, therefore, modulo

$$R = H \cdot \omega_p$$

To determine the direction of the vector R of the gyroscopic moment, you can use the L. Foucault rule (the rule of the same name parallelism of the axes of rotation). This rule for a rapidly rotating gyroscope is as follows: the vector of the gyroscopic moment R is directed in such a way that it tends to turn the vector of the angular momentum H to the vector of the angular velocity of the precession or, which is the same, the vector Ω of its own rotation of the gyroscope to the angular velocity vector. This means that in the right coordinate system, the vector R is directed in the direction from which the rotation from the vector H to the angular velocity vector is performed along the shortest distance counterclockwise.

The occurrence of a gyroscopic moment, which balances the moment of the external force, explains the fact that the precessional motion of the gyroscope occurs at a constant angular velocity.

The significance of this unique property of the gyroscope can hardly be overestimated, bearing in mind that one of the most important tasks related to the control of the movement of various objects requires the existence of some reference (reference) coordinate system obtained autonomously for its solution. The specified system can be created and is being created on the basis of a gyroscope, since even in the presence of some residual harmful moments (determined by the current level of development of science and technology), the gyroscope changes its initial orientation incomparably more slowly compared to any other body that does not have its own natural rotation. As is known, under the action of the moment of external forces on a non-gyroscopic body, the latter rotates with angular acceleration.

Now that the essence of the gyroscopic moment is known, it is not difficult to explain the property of a two-degree gyroscope, the external manifestation of which was characterized earlier. Indeed, referring again to fig. 30 b, it is easy to establish that the interaction of the angular momentum H of a gyroscope directed along the X-X axis with the angular velocity of forced rotation around the vertical axis leads to the appearance of a gyroscopic moment Rdirected along the Y-Y axis. In the general case, gyroscopic the moment is determined by the formula

$$R = H \cdot \omega \cdot \sin(H; \omega)$$

The gyroscopic moment rotates the ring together with the rotor until the angle between the vectors H and the angular velocity ω becomes zero.

Similarly, the occurrence of a gyroscopic moment explains the pressure exerted by a rotating body on the supports when the base in which they are fixed rotates. Knowing the gyroscopic moment and the distance between the supports, it is easy to calculate the load on the supports.

CONTROL QUESTIONS

1 Dynamics (definition). Galileo-Newton laws of mechanics.

2 Problems of dynamics for a free point.

3 Problems of dynamics for a non-free point.

4 Differential equations of motion of a material point in vector form.

5 Differential equations of motion of a material point in the coordinate system form.

6 Differential equations of motion of a material point in the natural form.

7 Solving differential equations of rectilinear motion of a point (F=const).

8 Solving differential equations of rectilinear motion of a point (F=f(t)).

9 Solving differential equations of rectilinear motion of a point (F=f(V)).

10 Solving differential equations of rectilinear motion of a point (F_x=f(x)).

11 Free oscillations of the point. Restorative power. Differential equations without frequency and period of oscillations.

12 Oscillations of a point with a resistance proportional to the speed (differential equation, characteristic equation, period of damped oscillations).

13 The dependence of the solution of differential equations of damped from the form of the roots of the characteristic equation.

14 Forced oscillations. Resonance.

15 The relative state of rest on the Earth's surface. Gravity.

16 Equation of relative motion of a point. Transfer and Coriolis forces of inertia points

17 Individual cases of relative motion of a point.

18 The principle of relativity of classical mechanics.

19 Mechanical system. Examples.

20 External and internal forces.

21 Properties of internal forces.

22 Mass of the system. The center of mass of the system.

23 Radius vector of the center of mass of the system.

24 Differential equations of motion of a system of points.

25 Moment of inertia of a homogeneous rectangular plate.

26 Moment of inertia of a homogeneous ring.

27 Moment of inertia of a ring sector.

28 Moment of inertia of a uniform rod.

29 Radius of inertia of the body.

30 Moment of inertia relative to coordinate axes.

31 Polar moment of inertia.

32 Moment of inertia of the body relative to the axis.

33 Moment of inertia of a uniform sphere.

34 Moment of inertia of a homogeneous cylindrical body.

35 Moment of inertia of a thin cylindrical shell.

36 Integral determination of the moment of inertia of a body relative to an axis.

37 Moment of inertia of the body relative to parallel axes.

38 Moment of inertia of a uniform thin disk.

39 Moment of inertia of a homogeneous cone.

40 Determination of the elementary work of the force applied to the point (a vector method of specifying the movement of a point).

41 Determination of the elementary work of the force applied to the point (a natural way of setting the motion of a point).

42 Determination of the elementary work of the force applied to the point (coordinate method of specifying the movement of a point).

43 The work of the force on the final displacement.

44 Potential force (definition).

45 The work of gravity.

46 Work of elastic force.

47 Work of gravity.

48 Strength of force.

49 Work of sliding friction force.

50 Work done by a force applied to a rotating body.

51 Work of the moment of a pair of forces.

52 Torque power.

53 The work of the force of rolling friction.

54 Work of potential forces.

55 Determination of power in the coordinate method of motion setting.

56 Measures of motion of a material point.

57 The amount of motion of a point and system (definition).

58 Elemental impulse of force.

59 Force impulse over a finite time interval.

60 Define the term "gyroscope".

61 List the components of the gyroscope.

62 What types of suspensions are used in gyroscopes?

63 Expand the essence of the three main properties of the gyroscope.

64 Formulate a theorem on the angular momentum of a gyroscope.

65 Formulate the Resal theorem.

66 Using the theorem "about the angular momentum" prove the basic properties of the gyroscope.

67 Explain the essence of the concept of "gyroscopic moment".

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