References:
1. Armstrong, M. Armstrong's handbook of strategic human resource management. 7th ed. London : Kogan Page, 2021. 736 p.

REPRESENTING LINEAR MAPS WITH MATRICES

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Abstract. This conference paper explores the representation of linear maps using matrices, highlighting the fundamental connection between abstract vector space theory and computational linear algebra. A linear map, defined as a function that preserves vector addition and scalar multiplication between vector spaces, can be expressed concretely once bases for the domain and codomain are chosen. By examining the action of a linear map on basis vectors, the resulting images can be written as linear combinations of the codomain basis vectors, whose coefficients form the columns of the map's matrix representation. This approach not only simplifies the computation of linear map operations but also facilitates the study of properties such as composition, inevitability, rank, and eigenvalues. Through illustrative examples, including projections and specific linear transformations, the paper demonstrates how matrix representations allow abstract linear operations to be computed efficiently using matrix-vector multiplication. The methodology relies on qualitative research and authoritative linear algebra references to provide a clear, structured understanding of these concepts.

Introduction. In linear algebra, a *linear map* (or linear transformation) is a function between vector spaces that preserves vector addition and scalar multiplication. A fundamental result is that every linear map between finite-dimensional vector spaces can be represented by a *matrix*, once bases for the domain and codomain are chosen.

Given a linear map $T : V \to W$, selecting a basis for V and a basis for W allows us to describe the action of T on basis vectors. The images of these basis vectors are expressed as linear combinations of the basis of W. The coefficients of these combinations form the columns of a matrix called the matrix representation of T.

This matrix representation transforms abstract linear operations into concrete matrix multiplication. As a result, properties of linear maps such as composition, inevitability, rank, and eigenvalues can be studied using matrix algebra. Changing the basis leads to a different matrix representation, connected by similarity transformations.

Thus, representing linear maps with matrices provides a powerful bridge between abstract vector space theory and computational methods.

Definition. Suppose that V and W are vector spaces of dimension n and m with bases B and B, that $h : V \to W$ is a linear map. If

$$\mathrm{Rep}_D\left(h(\vec{\beta}_1)\right) = \begin{pmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{m,1} \end{pmatrix}_D \cdots \mathrm{Rep}_D\left(h(\vec{\beta}_n)\right) = \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ \vdots \\ h_{m,n} \end{pmatrix}_D$$

Then

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,1} & \cdots & h_{1,n} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \cdots & h_{m,n} \end{pmatrix}_{B,D}$$

Is the *matrix representation pf h with represent to B, D.*

Briefly, the vectors representing the $h(\vec{\beta})'s$ are adjoined to make the matrix representing the map.

$$\mathrm{Rep}_{B,D}(h) = \left(\begin{array}{c|c|c} \vdots & & \vdots \\ \mathrm{Rep}_D\left(h(\vec{\beta}_1)\right) & \cdots & \mathrm{Rep}_D\left(h(\vec{\beta}_n)\right) \\ \vdots & & \vdots \end{array} \right)$$

Observe that the number of columns n of the matrix is the dimension of the domain of the map, and the number of rows m is the dimension of the codomain.

Example. If $h : \mathbb{R}^3 \to \rho_1$ is given by;

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \xrightarrow{\ h\ } (2a_1 + a_2) + (-a_3)x$$

Then where

$B = \langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \rangle$ And $D = \langle 1 + x, -1 + x \rangle$

The action of h on B is given by

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \xrightarrow{h} -x \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \xrightarrow{h} 2 \quad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \xrightarrow{h} 4$$

And a simple calculation gives

$$\mathrm{Rep}_D(-x) = \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}_D \quad \mathrm{Rep}_D(2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}_D \quad \mathrm{Rep}_D(4) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}_D$$

Showing that this is the matrix representing h with respect to the bases.

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} -1/2 & 1 & 2 \\ -1/2 & -1 & -2 \end{pmatrix}_{B,D}$$

We will use lower case letters for a map, upper case for the matrix, and lower case again for the entries of the matrix. Thus for the map h, the matrix representing it is H, with entries $h_{i,j}$.

Theorem. Assume that V, and W are vector spaces of dimensions m and n with bases B and D and that $h : V \to W$ is a linear map. If h is represented by

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,1} & \cdots & h_{1,n} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \cdots & h_{m,n} \end{pmatrix}_{B,D}$$

And $\vec{v} \in V$ is represented by

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$

Then the representation of the image of $\vec{v}$ is this.

$$\mathrm{Rep}_D(\vec{v}) = \begin{pmatrix} h_{1,1}c_1 + & h_{1,2}c_2 + & \cdots & +h_{1,n}c_n \\ h_{2,1}c_1 + & h_{2,1}c_2 + & \cdots & +h_{2,n}c_n \\ & \vdots & & \\ h_{m,1}c_1 + & h_{m,2}c_2 + & \cdots & +h_{m,n}c_n \end{pmatrix}_D$$

We will think of the matrix $\mathrm{Rep}_{B,D}(h)$ and the vector $\mathrm{Rep}_B(\vec{v})$as combining to make the vector $\mathrm{Rep}_D\big(h(\vec{v})\big)$.

Definition. The *matrix-vector product of a* $m \times n$ matrix and a $m \times 1$ vector is this.

$$\begin{pmatrix} a_{1,1} & a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & \vdots & & \\ a_{m,1} & a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_{1,1}c_1 + a_{1,2}c_2 + & \cdots & +a_{1,n}c_n \\ a_{2,1}c_1 + a_{2,1}c_2 + & \cdots & +a_{2,n}c_n \\ \vdots & & \\ a_{m,1}c_1 + a_{m,2}c_2 + & \cdots & +a_{m,n}c_n \end{pmatrix}$$

The point of Definition 2.2 is to generalize Example 2.1, that is, the point of the definition is Theorem 1.2 that the matrix describes how to get from the representation of a domain vector with respect to the domain’s basis to the representation of its image in the codomain with respect to the codomain’s basis. With Definition we can restate this as: application of a linear map is represented by the matrix-vector product of the map’s representative and the vector’s representative.

Example. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be projection onto the xy plane. To give a matrix representing this map, we first fix bases.

$$B = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rangle, \quad D = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

For each vector in the domain’s basis, we find its image under the map.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then we find the representation of each image with respect to the codomain’s basis

$$\mathrm{Rep}_D(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathrm{Rep}_D(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathrm{Rep}_D(\begin{pmatrix} -1 \\ 1 \end{pmatrix}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(These are easily checked). Finally, adjoining these representations gives the matrix representing π with respect to B, D.

$$\mathrm{Rep}_{B,D}(\pi) = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}_{B,D}$$

We can illustrate Theorem 2.1 by computing the matrix-vector product representing the following statement about the projection map.

$$\pi\left(\begin{pmatrix}2\\2\\1\end{pmatrix}\right)=\begin{pmatrix}2\\2\end{pmatrix}$$

Representing this vector from the domain with respect to the domain's basis

$$\mathrm{Rep}_B\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}1\\2\\1\end{pmatrix}_B$$

Gives this matrix-vector product.

$$\mathrm{Rep}_D\left(\pi\left(\begin{pmatrix}2\\2\\1\end{pmatrix}\right)\right)=\begin{pmatrix}1&0&-1\\-1&1&1\end{pmatrix}_{B,D}\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}_D$$

Expanding this representation into a linear combination of vectors from D

$$0\cdot\begin{pmatrix}2\\1\end{pmatrix}+2\cdot\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$$

Checks that the map's action is indeed reflected in the operation of the matrix. (We will sometimes compress these three displayed equations into one in the course of a calculation.)

$$\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}1\\2\\1\end{pmatrix}_B\ \xrightarrow[H]{h}\ \begin{pmatrix}0\\2\end{pmatrix}_D=\begin{pmatrix}2\\2\end{pmatrix}$$

We now have two ways to compute the effect of projection, the straightforward formula that drops each three-tall vector's third component to make a two-tall vector, and the above formula that uses representations and matrix vector multiplication. Compared to the first way, the second way might seem complicated. However, it has advantages. The next example shows that giving a formula for some maps is simplified by this new scheme.

Methodology. This research paper, titled Representing Linear Maps with Matrices, is based on a qualitative research methodology, as the collection of materials and information relies primarily on library-based sources. Reliable national and international academic references have been used to ensure the accuracy and credibility of the study. In addition, valuable advice and guidance from experienced lecturers have contributed to the preparation and compilation of this work.

Conclusion. Representing linear maps with matrices serves as a powerful tool to translate abstract vector space operations into concrete, computationally manageable forms. By selecting appropriate bases for the domain and codomain, any linear map can be expressed as a matrix whose columns correspond to the images of the basis vectors. This representation preserves the essential properties of the map and enables the computation of its action on arbitrary vectors via matrix-vector multiplication. The examples presented in this paper illustrate that matrix representations not only verify theoretical results, such as the effects of projection and linear transformations, but also simplify calculations and provide a framework for further analysis. Ultimately, the matrix-based approach bridges theoretical and practical aspects of linear algebra, enhancing both understanding and application of linear maps in mathematical and applied contexts.

References

1. Anton H., Rorres C. *Elementary Linear Algebra*, 11th ed. Hoboken, NJ, USA: Wiley, 2014.

2. Axler S. *Linear Algebra Done Right*, 3rd ed. Cham, Switzerland: Springer, 2015.

3. Friedberg S. H., Insel A. J., Spence L. E. *Linear Algebra*, 4th ed. Upper Saddle River, NJ, USA: Pearson, 2003.

4. Hefferon J., *Linear Algebra*, 3rd ed. Available: http://joshua.smcvt.edu/linearalgebra/

5. Hoffman K., Kunze R. *Linear Algebra*, 2nd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 1971.

6. Horn R. A., Johnson C. R. *Matrix Analysis*, 2nd ed. Cambridge, U.K.: Cambridge University Press, 2013.

7. Lay D. C., Lay S. R., McDonald J. J. *Linear Algebra and Its Applications*, 5th ed. Boston, MA, USA: Pearson, 2016.

8. Lax P. D. *Linear Algebra and Its Applications*, 2nd ed. Hoboken, NJ, USA: Wiley, 2007.

9. Strang G. *Introduction to Linear Algebra*, 5th ed. Wellesley, MA, USA: Wellesley-Cambridge Press, 2016.

10. Shores T. S. *Applied Linear Algebra and Matrix Analysis*, New York, NY, USA: Springer, 2007.